Tight Frames with Maximum Vanishing Moments and Minimum Support

Charles K. Chui, Wenjie He, and Joachim Stöckler

Abstract. The introduction of vanishing moment recovery (VMR) functions in our recent work (also called "fundamental functions" in an independent paper by Daubechies, Han, Ron, and Shen) modifies the so-called "unitary extension principle" to allow the construction of compactly supported affine frames with any desirable order of vanishing moments up to the order of polynomial reproduction of the given associated compactly supported scaling function. The objective of this paper is to unify and extend certain tight-frame results in the two papers mentioned above, with primary focus on the investigation of tight frame generators with minimum supports. In particular, a computational scheme to be described as an algorithm is developed for constructing such minimum-supported tight frame generators. An example is included as an illustration of this algorithm.

§1. Introduction

The parametric representation of curves and surfaces in terms of B-splines, and more generally NURBS, is a standard method in computer-aided design and manufacturing (CAD/CAM). Local support, variation-diminishing properties, and fast computational methods for B-splines and NURBS constitute some of the most important features for the selection of these basis functions for the CAD/CAM industry standards. However, only during the past 15 years, the properties of B-spline multiresolution analysis and spline wavelets entered into the picture, and it was clear from the very beginning that there does not exist an L_2-orthonormal basis of continuous spline-wavelets with compact support. This led to the idea of using a semi-orthogonal spline-wavelet basis with local support for the synthesis (the basis functions of the parametric representation) and dual basis functions of global support, but exponential decay, for the analysis (the dual functionals rendering the coefficients in the representation),

Approximation Theory X: Abstract and Classical Analysis
Charles K. Chui, Larry L. Schumaker, and Joachim Stöckler (eds.), pp. 1–20.

ISBN 0-8265-1415-4.

although for better performance, change-of-bases is recommended in order to use the same compactly supported spline-wavelets both for analysis and for synthesis (see [1]). Of course, semi-orthogonal wavelets and their duals have the maximum order of vanishing moments in the sense that they annihilate all polynomials of degree $L-1$, when L^{th} order B-spline functions are used to construct the spline-wavelets.

To avoid the need of change-of-bases but still use compactly supported spline-wavelets for both analysis and synthesis, a more general approach to multiresolution representation of curves and surfaces is offered by tight affine frames of $L_2 = L_2(\mathbb{R})$, which also lead to stable parametric representations. In our present work, we discuss the shift-invariant setting in L_2. Extension to bounded intervals where B-splines with non-uniform knot sequences provide a local basis, is studied in a paper under preparation.

For the L^2 setting, tight affine frames are generated by functions $\psi_i \in L_2$ by shifts and dilations, such that the family

$$\Psi := \{\psi_{i;j,k} = 2^{j/2}\psi_i(2^j \cdot -k);\ j \in \mathbb{Z},\ k \in \mathbb{Z},\ 1 \le i \le n\}$$

satisfies

$$A\|f\|^2 = \sum_{i=1}^{n} \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{i;j,k}\rangle|^2, \qquad f \in L_2(\mathbb{R}), \tag{1}$$

where the constant $A > 0$, called the tight frame constant, does not depend on f. If $A = 1$ we say that Ψ is a normalized tight frame. Here, normalization is achieved simply by dividing each ψ_i by $A^{1/2}$. In order to simplify notation, we also call the functions $\{\psi_i : 1 \le i \le n\}$ tight frame generators of Ψ. If the functions ψ_i are finite linear combinations of B-splines, they can be employed as the multiresolution synthesis and analysis tool for parametric spline curves and tensor-product surfaces. In addition to having local support, the tight frame generators should also exhibit L vanishing moments for the purpose of providing an effective analysis tool.

In the following, we consider a more general setting in our discussion of the construction and characterization of tight frame generators. Let $\phi \in L_2$ be a refinable function with compact support, satisfying a refinement equation

$$\phi(x) = \sum_{k=M}^{N} p_k \phi(2x-k), \tag{2}$$

where p_k are real coefficients and ϕ is the normalized solution of (2) with $\hat{\phi}(0) = 1$ such that the corresponding Laurent polynomial

$$P(z) = \frac{1}{2}\sum_{k=M}^{N} p_k z^k, \tag{3}$$

called the two-scale symbol of ϕ, satisfies $P(-1) = 0$. The integer $L \geq 1$ is so chosen that $P(z) = (1+z)^L P_0(z)$, and P_0 is a Laurent polynomial with $P_0(-1) \neq 0$. It is known [2,7] that the spaces V_j that are the closure of the span of the $2^{-j}\mathbb{Z}$-shifts of $\phi(2^j\cdot)$, generate a multiresolution approximation

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \rightarrow L_2(\mathbb{R}).$$

Without giving away much generality, we assume throughout that ϕ is a minimally supported refinable function in V_0, and therefore by the results in [8], the integer shifts of ϕ constitute a Riesz basis of V_0. (Note that this conclusion does not generalize to higher dimensions.) The characterization and construction of (minimally supported) tight frame generators $\psi_i \in V_1$, $1 \leq i \leq n$, that have L_0 vanishing moments, where $1 \leq L_0 \leq L$ is arbitrary, is our main concern in this paper. The functions ψ_i can be defined by

$$\hat{\psi}_i(\omega) = Q_i(z)\hat{\phi}(\omega/2), \qquad z := e^{-i\omega/2},$$

where Q_i, $1 \leq i \leq n$, are Laurent polynomials with real coefficients such that

$$Q_i(z) = (1-z)^{L_0} q_i(z), \qquad 1 \leq i \leq n, \tag{4}$$

with new Laurent polynomials q_i. The factor $(1-z)^{L_0}$ entails the vanishing moment property of ψ_i.

A characterization of all orthonormal wavelets ψ (without the necessity of being defined by an MRA) appears in the book [6], and its extention to tight frame generators $\{\psi_i\}$ was given later by others such as [5,12]. Tight frame generators in the given setting of an MRA (i.e. associated with a compactly supported scaling function) can be characterized by the identities

$$S(z^2)P(z)P(1/z) + \sum_{i=1}^{N} Q_i(z)Q_i(1/z) = S(z), \tag{5}$$

$$S(z^2)P(z)P(-1/z) + \sum_{i=1}^{N} Q_i(z)Q_i(-1/z) = 0, \tag{6}$$

which must be satisfied for all $z \in \mathbb{C} \setminus \{0\}$. Here, the function S must be a Laurent polynomial with $S(1) = 1$ and $S(z) \geq 0$ on the unit circle (to be denoted by $\mathbb{T}$ throughout this paper). Moreover, the functions ψ_i have vanishing moments of order L_0, *i.e.*,

$$\int_{\mathbb{R}} x^\ell \psi_i(x)dx = 0, \qquad 0 \leq \ell < L_0, \tag{7}$$

if and only if

$$S(z)\Phi(z) - 1 = \mathcal{O}(|z-1|^{2L_0}) \qquad \text{near } z = 1, \tag{8}$$

where $\Phi(z) = \sum_k c_k z^k$ is the autocorrelation symbol of ϕ defined by the coefficients $c_k = \int_{\mathbb{R}} \phi(x)\phi(x+k)dx$. The last two statements were mostly developed in [2], [4]. Minor improvements for tight frames are made in the present paper in Theorems 1 and 2. We coined S as a vanishing moment recovery (VMR) function in [2]; the term "fundamental function" is employed in [4]. Well-known properties of the autocorrelation function of cardinal B-splines show that $L_0 > 1$ cannot be achieved, if ϕ is a cardinal B-spline and S in (5)–(6) is the constant function 1. This was implicitly assumed in the construction of tight frame generators in [12], however, and one of the tight frame generators of their construction by a matrix extension method has only one vanishing moment.

The outline of this paper is as follows. A review of some results in [2], [4] with minor improvement is given in Section 2, where the characterization by (5)–(8) is developed. The natural question that addresses to whether or not for any refinable (stable) function ϕ, there exists a VMR function S such that $L_0 = L$ vanishing moments can be obtained for all compactly supported generators ψ_i, was answered to be positive in [2] (see Theorem 3). In Section 3 we extend the technique in [2] for the construction of tight frame generators with two functions ψ_1, ψ_2, in order to find all such generators that have minimum support. Although the derivation looks rather technical, the outcome is a simple algorithm which is given at the end of Section 3.

§2. Results for Tight Frames

In this section, we restrict our discussion to the results of the authors on univariate tight frames in [2] and of Daubechies et al. in [4], particularly in characterization and construction of tight frame generators.

Theorem 1. *Let ϕ be a refinable function with compact support and two-scale symbol P as in (3), and assume that the shifts of ϕ are stable. Let Q_i, $1 \le i \le n$, be Laurent polynomials with real coefficients. Then the functions ψ_i defined by $\hat{\psi}_i(2\xi) = Q_i(e^{-i\xi})\hat{\phi}(\xi)$ are tight frame generators, with frame constant* 1, *if and only if there exists a Laurent polynomial S with real coefficients, $S(1) = 1$, $S(z) \ge 0$ on $\mathbb{T}$, that satisfies (5)–(6).*

Theorem 1 is a reformulation of a result in [2], under the additional hypothesis of stability of the shifts of ϕ. Note that identity (5) implies that $Q_i(1) = 0$ holds for all $1 \le i \le n$. This condition is necessary and sufficient for the boundedness of the series on the right-hand side of (1).

The characterization of tight frame generators with L_0 vanishing moments in (7) can also be given. The sufficiency of property (8) of the

Laurent polynomial S was shown in [2]. Our subsequent discussion in the present paper shows that (8) is also necessary.

Theorem 2. *Let $1 \le L_0 \le L$ and let the assumptions of Theorem 1 on ϕ and Q_i, $1 \le i \le N$, be satisfied. Then the functions ψ_i are tight frame generators, with frame constant 1 and L_0 vanishing moments, if and only if there exists a Laurent polynomial S with real coefficients, $S(1) = 1$, $S(z) \ge 0$ on $\mathbb{T}$, that satisfies (5)–(6) and (8).*

Proof: We recall from [1] that the autocorrelation symbol Φ satisfies $\Phi(1) = 1$ and

$$|P(z)|^2\Phi(z) + |P(-z)|^2\Phi(-z) = \Phi(z^2), \quad z \in \mathbb{T}. \tag{9}$$

Consequently, we obtain

$$\frac{1}{\Phi(z)} - \frac{|P(z)|^2}{\Phi(z^2)} = \mathcal{O}(|z-1|^{2L}) \qquad \text{near } z = 1. \tag{10}$$

Let $1 \le L_0 \le L$, where L is the multiplicity of the zero $z = -1$ of P. All Laurent polynomials Q_i, $1 \le i \le n$, have the form (4), as a consequence of (5), if and only if

$$S(z) - S(z^2)|P(z)|^2 = \mathcal{O}(|z-1|^{2L_0}) \qquad \text{near } z = 1. \tag{11}$$

If we insert (10), we obtain the equivalent relation

$$S(z) - \frac{1}{\Phi(z)} - |P(z)|^2\left[S(z^2) - \frac{1}{\Phi(z^2)}\right] = \mathcal{O}(|z-1|^{2L_0}).$$

By analyticity of S and Φ near 1 and $P(1) = 1$, the previous relation is equivalent to

$$S(z) - \frac{1}{\Phi(z)} = \mathcal{O}(|z-1|^{2L_0}),$$

which, in turn, is equivalent to (8). If we combine this result with Theorem 1, we obtain the result of Theorem 2. $\square$

There are many ways to find a Laurent polynomial S that satisfies (8). By symmetry of $\Phi(e^{i\xi})$ around 0, there exists a unique solution S of (8) that has the form

$$S(z) = \sum_{k=0}^{L_0-1} s_k(z^k + z^{-k}). \tag{12}$$

The coefficients s_k of this solution are real, so that S is real on the unit circle and has minimum degree (possibly less than $2(L_0-1)$). The function

S defined in this way may not be a VMR function, however. In addition to (8), S must be chosen such that identities (5)–(6) admit Laurent polynomial solutions for Q_i, $1 \le i \le n$. By rewriting these identities with an argument $-z$ for z, we have the equivalent condition

$$\begin{aligned}\mathcal{M}(z) &:= \begin{pmatrix} S(z) - S(z^2)P(z)P(1/z) & -S(z^2)P(1/z)P(-z) \\ -S(z^2)P(z)P(-1/z) & S(-z) - S(z^2)P(-z)P(-1/z) \end{pmatrix} \\ &= \begin{pmatrix} Q_1(1/z) & \cdots & Q_n(1/z) \\ Q_1(-1/z) & \cdots & Q_n(-1/z) \end{pmatrix} \begin{pmatrix} Q_1(z) & Q_1(-z) \\ \vdots & \vdots \\ Q_n(z) & Q_n(-z) \end{pmatrix}, \end{aligned} \tag{13}$$

which we use as a substitute for (5)–(6) from now on. Apparently, the VMR function S must be so chosen that, in addition to (8), the matrix $\mathcal{M}(z)$ is positive semi-definite for all $z \in \mathbb{T}$. This is accomplished in [2] (see Theorem 5) as follows.

Theorem 3. *Let the assumptions on ϕ in Theorem 1 be satisfied. Then there exists a Laurent polynomial S with real coefficients, $S(1) = 1$, $S(z) > 0$ for all $z \in \mathbb{T}$, such that (8) is satisfied and $\mathcal{M}(z)$ in (13) is positive semidefinite for all $z \in \mathbb{T}$.*

We only mention the main ideas of the proof of this result. First, we make use of spectral properties of the transfer operator

$$T_{|P|^2}(f)(z^2) = |P(z)|^2 f(z) + |P(-z)|^2 f(-z), \qquad z \in \mathbb{T},$$

which are developed in [9]. This leads to the construction of a Laurent polynomial R that is an eigenfunction of $T_{|P|^2}$ with respect to an eigenvalue $0 < \lambda < 1$, satisfies $R = \mathcal{O}(|1-z|^{2L_0})$ and $R(z) > 0$ on $\mathbb{T} \setminus \{1\}$. The existence of R, which is not shown in [9], can be deduced from a study of positivity and irreducibility of the restriction of $T_{|P|^2}$ to certain invariant subspaces. The notations in [11] prove to be very useful for this discussion. Then, we construct a Laurent polynomial S with real coefficients, for which

$$\frac{1}{\Phi + \beta R} \le S \le \frac{1}{\Phi + R}, \qquad 1 < \beta < 1/\lambda,$$

is satisfied on $\mathbb{T}$. This problem is solved by trigonometric approximation with interpolatory constraints at $z = 1$. The upper and lower bounds for $1/S$ are inserted in

$$T_{|P|^2}(1/S) \le T_{|P|^2}(\Phi + \beta R) = \Phi + \beta\lambda R \le \Phi + R \le 1/S, \tag{14}$$

where the equality sign is justified by (9). Finally, the matrix $\mathcal{M}$ is shown to be positive semi-definite on $\mathbb{T}$, which is a consequence of

$$\frac{S(z) - S(z^2)|P(z)|^2}{S(z)S(z^2)} \ge (\mathrm{id} - T_{|P|^2})(1/S) = \frac{\det \mathcal{M}(z^2)}{S(z)S(-z)S(z^2)},$$

where the term in the middle is nonnegative by (14).

A similar result as stated in Theorem 3 for refinable functions ϕ with dilation factor $M \geq 2$ was recently obtained in our work [3]. The proof does not provide precise bounds for the degree of S. For the special case of cardinal B-splines, however, a precise analysis of the minimum degree Laurent polynomial S in (12) is given in [4]. Moreover, it can be shown that their result implies that the matrix $\mathcal{M}$ in (13) is positive semi-definite on $\mathbb{T}$. The same result is confirmed for low order B-splines ($L \leq 4$) in [2].

§3. Tight Frames with Two Generators and Minimum Support

From now on we assume that the VMR function S is given such that (8) holds and the matrix $\mathcal{M}$ in (13) is positive semi-definite on $\mathbb{T}$. Moreover, S is supposed to have minimum degree, whenever we discuss factorizations of $\mathcal{M}$ with minimum degree Laurent polynomials. We show, after performing three consecutive transformations (15), (21), and (27) of this matrix, that a factorization of $\mathcal{M}$ in (13) exists where we need only two Laurent polynomials Q_1 and Q_2. Upper bounds for the degree of Q_1 and Q_2 can be derived from Theorem 4. These bounds are experimentally found to be sharp for tight frame generators from B-spline MRA of low order. In this way, tight frame generators $\psi_1, \psi_2 \in V_1$ with L_0 vanishing moments are constructed. Special emphasis is given to the aspect of finding minimum degree Laurent polynomials Q_1, Q_2 in (13), because they define tight frame generators with minimum support. This is the reason why we keep track of the degree of all Laurent polynomials that are involved in the construction, and why certain transformations of $\mathcal{M}$ are chosen. Although this produces some notational overhead, there is a simple algorithm at the end of this section which summarizes all steps for the construction of tight frame generators with minimum support in a compact way.

Before we begin with the construction, we need to agree on the meaning of comparing size of support of function pairs (ψ_1, ψ_2). In the following, $|I|$ denotes the length of an interval I. For a function f on $\mathbb{R}$, we let $I(f)$ be the smallest interval that contains the support of f. We define a partial ordering as follows. The support of the pair (ψ_1, ψ_2) is larger than the support of $(\tilde{\psi}_1, \tilde{\psi}_2)$ if one of the two conditions is satisfied: (1) $|I(\psi_1)| > |I(\tilde{\psi}_1)|$ and $|I(\psi_2)| \geq |I(\tilde{\psi}_2)|$, or (2) $|I(\psi_1)| = |I(\tilde{\psi}_1)|$ and $|I(\psi_2)| > |I(\tilde{\psi}_2)|$. Similarly, we define a partial ordering for pairs of Laurent polynomials. The pair (Q_1, Q_2) has larger degree than the pair $(\tilde{Q}_1, \tilde{Q}_2)$ if either $\deg(Q_1) > \deg(\tilde{Q}_1)$ and $\deg(Q_2) \geq \deg(\tilde{Q}_2)$, or $\deg(Q_1) = \deg(\tilde{Q}_1)$ and $\deg(Q_2) > \deg(\tilde{Q}_2)$. Here, the degree of a Laurent polynomial $Q(z) = \sum_{k=m}^{n} a_k z^k$ is defined to be $n - m$, if $a_m a_n \neq 0$. The space of all Laurent polynomials Q as above is denoted by $L[m:n]$. First we recall some important definitions and agree on the notations. The two-scale symbol of ϕ is given by

$$P(z) = \left(\frac{1+z}{2}\right)^L P_0(z) = \sum_{k=-M_P}^{N_P} p_k z^k, \qquad p_{-M_P} p_{N_P} \neq 0.$$

The VMR function S is a Laurent polynomial with real coefficients, which is real on $\mathbb{T}$. Therefore, it has the expansion

$$S(z) = \sum_{k=0}^{N_S} s_k (z^k + z^{-k}), \qquad s_{N_S} \neq 0.$$

The first transformation of the matrix $\mathcal{M}$ is performed by making use of (4) in order to replace identity (13) with

$$\begin{aligned} \mathcal{M}_0(z) &= \begin{pmatrix} X(z) & Y(z) \\ Y(-z) & X(-z) \end{pmatrix} \\ &= \begin{pmatrix} q_1(1/z) & q_2(1/z) \\ q_1(-1/z) & q_2(-1/z) \end{pmatrix} \begin{pmatrix} q_1(z) & q_1(-z) \\ q_2(z) & q_2(-z) \end{pmatrix}, \end{aligned} \tag{15}$$

where q_1, q_2 are (minimum degree) Laurent polynomials and

$$\begin{aligned} X(z) &:= \frac{S(z) - S(z^2)P(z)P(1/z)}{(1-z)^{L_0}(1-1/z)^{L_0}}, \\ Y(z) &:= \frac{-S(z^2)P(1/z)P(-z)}{(1+z)^{L_0}(1-1/z)^{L_0}}. \end{aligned} \tag{16}$$

By the assumptions on S and P, the functions X and Y are Laurent polynomials that have the form

$$X(z) = \sum_{k=0}^{N_X} x_k (z^k + z^{-k}), \qquad Y(z) = \sum_{k=0}^{N_X} y_k (z^k + (-1)^k z^{-k}), \tag{17}$$

where $N_X = 2N_S + M_P + N_P - L_0$ and the leading coefficients satisfy $x_{N_X} = \pm y_{N_X} \neq 0$.

The determinant

$$\det \mathcal{M}_0(z) = \frac{S(z)S(-z) - S(z^2)(S(-z)P(z)P(1/z) + S(z)P(-z)P(-1/z))}{((1-z)(1-1/z)(1+z)(1+1/z))^{L_0}}$$

is an even Laurent polynomial which is nonnegative on $\mathbb{T}$. We assume that it does not vanish identically. Then we define

$$\Delta(z) = \sum_{k=0}^{N_\Delta} \delta_k (z^k + z^{-k}), \quad \delta_{N_\Delta} \neq 0, \tag{18}$$

such that $4\Delta(z^2) = \det \mathcal{M}_0(z)$. Note that

$$N_X - N_\Delta \geq N_S + M_P + N_P - K \geq \begin{cases} 1, & \text{if } N_S + M_P + N_P < 3, \\ 2, & \text{if } N_S + M_P + N_P \geq 3, \end{cases} \tag{19}$$

where K is the largest integer such that the coefficient of z^{2K} in the expansion of $S(-z)P(z)P(1/z) + S(z)P(-z)P(-1/z)$ is nonzero. (If this Laurent polynomial equals zero, we let $K = 0$.)

It is suitable to present the following result at this point, although the proof of the existence can only be given later. The second part of the theorem is clear, since $X(z) = q_1(z)q_1(1/z) + q_2(z)q_2(1/z)$ by (15).

Theorem 4. *With X, Y, and Δ from above, let $\mu := \min(2, N_X - N_\Delta)$. Then there exist Laurent polynomials q_1, q_2 with real coefficients and* $\deg q_1 = N_X$, $\deg q_2 \leq N_X - \mu$ *which satisfy (15). Moreover, no pair of Laurent polynomials (q_1, q_2) with* $\deg q_1 < N_X$ *and* $\deg q_2 < N_X$ *can satisfy (15).*

We continue with our preparations for the construction of a factorization of $\mathcal{M}$, from which the existence part of Theorem 4 will be deduced. As the second transformation of $\mathcal{M}$, we perform a polyphase decomposition in order to decouple the entries of the matrix $\mathcal{M}_0$ in (15). If we let

$$\begin{aligned} &\nu = 0, \quad T(z) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}, \quad \text{if } x_{N_X} = y_{N_X}, \\ &\nu = 1, \quad T(z) = \frac{1}{2}\begin{pmatrix} 1/z & -1/z \\ 1 & 1 \end{pmatrix}, \quad \text{if } x_{N_X} = -y_{N_X}, \end{aligned} \tag{20}$$

we can define Laurent polynomials A, B, and C by

$$\begin{pmatrix} A(z^2) & B(z^2) \\ B(1/z^2) & C(z^2) \end{pmatrix} = T(z) \begin{pmatrix} X(z) & Y(z) \\ X(-z) & Y(-z) \end{pmatrix} T^t(1/z). \tag{21}$$

Then q_1, q_2 satisfy (15) if and only if

$$q_i(z) = z^{-\nu+2\ell_i}[u_i(z^2) + z v_i(z^2)], \qquad i = 1, 2, \tag{22}$$

where ℓ_i is an integer, the lowest order monomial of $[u_i(z^2) + z v_i(z^2)]$ is either 1 or z, and

$$\begin{pmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{pmatrix} = \begin{pmatrix} u_1(1/z) & u_2(1/z) \\ v_1(1/z) & v_2(1/z) \end{pmatrix} \begin{pmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{pmatrix}. \tag{23}$$

We also note that Δ in (18) is the determinant of the matrix on the left-hand side of (23). Therefore, $d = u_1v_2 - u_2v_1$ satisfies the relation

$$d(z)d(1/z) = \Delta(z) = A(z)C(z) - B(z)B(1/z). \tag{24}$$

For later reference we display the representations of the new Laurent polynomials

$$\begin{aligned} A(z) &= \sum_{k=0}^{N_A} a_k(z^k + z^{-k}), \quad a_k = \frac{x_{2k} + (-1)^\nu y_{2k}}{2}, \\ C(z) &= \sum_{k=0}^{N_C} c_k(z^k + z^{-k}), \quad c_k = \frac{x_{2k} - (-1)^\nu y_{2k}}{2}, \\ B(z) &= \sum_{k=-M_B}^{N_B} b_k z^k, \quad b_k = \frac{x_{2k+1} - (-1)^\nu y_{2k+1}}{2}, \end{aligned} \tag{25}$$

where we let $y_{-k} = -y_k$ for odd k in the last expression, see (17). Here, N_A, N_C, M_B, and N_B are the largest integers for which the summands are nonzero. As a consequence of the choice of the parameter $\nu \in \{0,1\}$ in (20), we obtain that

$$\begin{aligned} &N_A = N_X/2, && M_B, N_B+1, N_C+1 \le N_X/2, && \text{if } N_X \text{ is even}, \\ &M_B = (N_X+1)/2, && N_A, N_C, N_B+1 \le (N_X-1)/2, && \text{if } N_X \text{ is odd}. \end{aligned} \tag{26}$$

The following observation summarizes the link between solutions of (15) and (23).

Proposition 5. *Let N_X be given as in (17), and let N be the largest integer less than or equal to $(N_X+1)/2$. If (q_1, q_2) is a solution of (15) with $\deg q_i \le N_X$, $i = 1, 2$, then the polyphase components u_1, u_2, v_1, v_2 of q_1 and q_2, as defined in (22), are elements of $L[0 : N]$ and define a factorization (23).*

Proof: If q_1 and q_2 satisfy the assumptions of the proposition, we obtain from (22) that $z^{\nu - 2\ell_i} q_i \in L[0 : N_X+1]$. Hence, the polyphase components u_i, v_i in (22), $i = 1, 2$, are elements of $L[0 : N]$. It is clear from the construction that they define a factorization (23). □

Remark. The value of the last proposition for the construction of tight frame generators with minimum support can be seen as follows. In order to find *all* minimum degree solutions (q_1, q_2) of (15), where $\deg q_i \le N_X$, $i = 1, 2$, the search should be extended over all polyphase components $u_i, v_i \in L[0 : N]$. An algorithm that simplifies this search to an elementary problem of linear algebra is developed next.

Our main task remains to show the existence of the factorization of the positive semi-definite matrix on the left-hand side of (23). We have developed in [2] a method for the factorization of such matrices under certain constraints on the Laurent polynomial entries. Therefore, we arrange for a "generic" form of the matrix by the following third transformation. This transformation turns out to be more suitable for the discussion of minimum degree factorizations than the transformation in Lemma 4 of [2].

Lemma 6. *Assume that there is no common zero in* $\mathbb{C} \setminus \{0\}$ *of all four Laurent polynomials* A, B, C, *and* $B(1/z)$. *Then for almost every* $r \in \mathbb{R}$, *the Laurent polynomials* $\widetilde{A}(z), \widetilde{B}(z)$ *defined by*

$$\begin{pmatrix} \widetilde{A}(z) & \widetilde{B}(z) \\ \widetilde{B}(1/z) & C(z) \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \tag{27}$$

have the form (25), with parameters $N_{\tilde{A}} = \max(N_A, N_C, M_B, N_B)$, $M_{\tilde{B}} = \max(M_B, N_C)$, $N_{\tilde{B}} = \max(N_B, N_C)$, *and* $\widetilde{A}, \widetilde{B}$ *have no common zeros in* $\mathbb{C} \setminus \{0\}$.

Proof: The determinant of the matrix on the left hand side of (27) is the same, for every $r \in \mathbb{R}$; indeed, it agrees with Δ in (18). Consequently, only the roots of Δ are possible candidates for common zeros of

$$\widetilde{A}(z) = A(z) + r(B(z) + B(1/z)) + r^2 C(z), \qquad \widetilde{B}(z) = B(z) + rC(z).$$

For a fixed root z_0 of Δ, there is at most one value of $r \in \mathbb{R}$ such that $\widetilde{A}(z_0) = \widetilde{B}(z_0) = 0$, except for the case where $A(z_0) = B(z_0) = B(1/z_0) = C(z_0) = 0$. This shows that for at most $2N_\Delta$ values of r, the Laurent polynomials $\widetilde{A}$ and $\widetilde{B}$ can have common zeros, given the assumption on A, B, and C of the lemma. Evidently, almost all values of r yield $N_{\tilde{A}} = \max(N_A, M_B, N_B, N_C)$. The other relations are clear as well. This completes the proof of the lemma. $\square$

Remark. It follows from the invertibility of T in (21) that all four Laurent polynomials A, B, $B(1/z)$, and C have a common zero $z_0 \neq 0$, if and only if $X(z_0) = X(-z_0) = Y(z_0) = Y(-z_0) = 0$. In the literature on stability of refinable functions, a common zero of X and $X(-z)$ is often called a symmetric zero of X. Therefore, the assumption of Lemma 6 is equivalent to the condition that X and Y have no common symmetric zeros. In our experiments we never encountered a case where this condition is violated.

We have now completed the three transformations of the matrix $\mathcal{M}$, that are the combination of (15), (21), and (27). Note that $N := N_{\tilde{A}}$ is the largest number among $\{N_A, M_B, N_B, N_C\}$. In view of (26), this is the largest integer less than or equal to $(N_X + 1)/2$. Since all transformations are invertible, we find a one-to-one correspondence between all Laurent polynomial solutions $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)$ of the factorization problem

$$\begin{pmatrix} \widetilde{A}(z) & \widetilde{B}(z) \\ \widetilde{B}(1/z) & C(z) \end{pmatrix} = \begin{pmatrix} \tilde{u}_1(1/z) & \tilde{u}_2(1/z) \\ \tilde{v}_1(1/z) & \tilde{v}_2(1/z) \end{pmatrix} \begin{pmatrix} \tilde{u}_1(z) & \tilde{v}_1(z) \\ \tilde{u}_2(z) & \tilde{v}_2(z) \end{pmatrix} \tag{28}$$

and the Laurent polynomial solutions (q_1, q_2) of (15), which gives

$$q_i(z) = z^{2\ell_i - \nu}(\tilde{u}_i(z^2) + (z-r)\tilde{v}_i(z^2)), \quad i = 1, 2. \tag{29}$$

If we compare (29) with (22) and apply Proposition 5, we obtain

$$\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2 \in L[0:N], \qquad N = N_{\tilde{A}}, \tag{30}$$

if q_1, q_2 have degree at most N_X. It is this "a priori" specification of the degree of the polynomials $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2$, which renders the following procedure for the construction of solutions feasible. Our main tool is a link between all factorizations of the type (28) under the constraints (30) and the system of equations

$$\widetilde{B}(z)\tilde{u}_1(z) - d(z)\tilde{u}_2(1/z) - \widetilde{A}(z)\tilde{v}_1(z) = 0, \tag{31}$$

$$\widetilde{B}(z)\tilde{u}_2(z) + d(z)\tilde{u}_1(1/z) - \widetilde{A}(z)\tilde{v}_2(z) = 0, \tag{32}$$

$$\tilde{u}_1^2(1) + \tilde{u}_2^2(1) = \widetilde{A}(1), \tag{33}$$

where d is a Laurent polynomial that satisfies (24). Recall that d can be viewed as a "square root" of the determinant Δ of the matrix on the left-hand side of (28). Note that (33) is merely a normalizing condition, as long as u_1 and u_2 do not vanish simultaneously at 1.

The following result was shown in Theorem 4 of [2]. In order to simplify notations, we leave out the tilde signs, which we then must do in (31)–(33) and (28) as well. The result is reformulated in order to expose the main tools needed for the proof of Theorem 4 and our new algorithm at the end of this section. The proof in [2] is based on arguments concerning elementary algebraic properties of Laurent polynomials and arguments from linear algebra and is omitted here.

Theorem 7. *Let* $\begin{pmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{pmatrix}$ *be a matrix of Laurent polynomials*

$$A, B, C \in L[-N:N], \quad A(z) = \sum_{k=0}^{N} a_k(z^k + z^{-k}) \quad \text{with } a_N \neq 0.$$

Suppose that the matrix is positive semi-definite on $\mathbb{T}$*, and the determinant* Δ *does not vanish identically. Then*

(i) *If polynomials $u_1, u_2, v_1, v_2 \in L[0:N]$ satisfy (28), then the equations (31)–(33), with $d := u_1 v_2 - u_2 v_1$, are satisfied.*

Let $d \in L[0:2N]$ be a polynomial that satisfies $d(z)d(1/z) = \Delta(z)$, and assume, in addition to the aforementioned assumptions, that A and B have no common zeros in $\mathbb{C} \setminus \{0\}$. Moreover,

(ii) *If $u_1, u_2, v_1 \in L[0:N]$ define a nontrivial solution of (31), then there exists a polynomial $v_2 \in L[0:N]$ such that u_1, u_2, v_2 is a nontrivial solution of (32).*

(iii) *If $u_1, u_2, v_1, v_2 \in L[0:N]$ define a nontrivial solution of (31)–(32), then there is a constant $c > 0$ such that cu_1, cu_2, cv_1, cv_2 define a factorization (28).*

Let us discuss the meaning of Theorem 7 for our construction of minimum degree solutions (q_1, q_2) of (15). Parts (i) and (iii) of the theorem, together with (30), confirm that it is sufficient to inspect all solutions of (31)–(33) that are polynomials in $L[0:N]$, in order to find all solutions (q_1, q_2) whose degree does not exceed N_X. The parameter d in the equations (31)–(32) must vary over all polynomials in $L[0:2N]$ which have real coefficients and constitute a root of the determinant Δ. Therefore, d has the form

$$d(z) = z^\ell \sum_{k=0}^{N_\Delta} d_k z^k, \tag{34}$$

where $0 \le \ell \le 2N - N_\Delta$ and the real coefficients d_k determine all possible solutions of $d(z)d(1/z) = \Delta(z)$. There exist many methods in order to obtain these polynomials, *e.g.* spectral factorization as proposed in [10].

The second part of Theorem 7 implies a considerable reduction of the amount of work for solving (31)–(32). The following method makes use of elementary linear algebra. We insert the unknown coefficients of $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2$, which we denote by $\tilde{u}_{1,k}$ etc., into two vectors

$$\begin{aligned} \vec{x} &= (\tilde{v}_{1,0}, \ldots, \tilde{v}_{1,N}, \tilde{u}_{1,0}, \tilde{u}_{2,0}, \ldots, \tilde{u}_{1,N}, \tilde{u}_{2,N})^t, \\ \vec{y} &= (\tilde{v}_{2,0}, \ldots, \tilde{v}_{2,N}, \tilde{u}_{2,0}, -\tilde{u}_{1,0}, \ldots, \tilde{u}_{2,N}, -\tilde{u}_{1,N})^t. \end{aligned} \tag{35}$$

Equation (31) can then be written as a linear homogeneous system $Z\vec{x} = 0$ by expanding the left hand side of (31) into a Laurent series and setting the coefficient of z^j, $-N \le j \le 2N$, to zero. Z is the corresponding real matrix with $3N+1$ rows and $3N+3$ columns. Likewise, equation (32) can be written as $Z\vec{y} = 0$ with the same matrix Z. Obviously, each of the systems $Z\vec{x} = 0$ and $Z\vec{y} = 0$ admits nontrivial solutions, since the number of unknowns in each system exceeds the number of equations by 2. Thanks to part (ii) of Theorem 7, the seemingly overdetermined system $(Z\vec{x} = 0,\ Z\vec{y} = 0)$, which has $6N+2$ equations for $4N+4$ unknowns, has as solutions all quadruplets $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2$ where $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1)$ is a solution of the system $Z\vec{x} = 0$. This effect can also be read off the reduced row echelon form of Z, which is part (i) of the following lemma. Other features such as zero entries of the solution vectors $\vec{x}$ and $\vec{y}$ can also be determined from simple properties of $\widetilde{A}$, $\widetilde{B}$, and d, see parts (ii) and (iii) of the lemma. For ease of notation we skip the tilde sign again.

Lemma 8. *Assume that the Laurent polynomials A, B, C in (25) and d in (34) are given where $N := N_A \geq \max(N_C, M_B, N_B)$ and $d \in L[\ell : \ell + N_\Delta] \subset L[0 : 2N]$. Furthermore, assume that A and B have no common zeros in $\mathbb{C} \setminus \{0\}$. Then the matrix Z from above has full rank. Its first $3N+1$ columns are linearly independent, and its reduced row echelon form is given by*

$$\begin{pmatrix} 1 & & & g_1 & h_1 \\ & \ddots & & & \\ & & 1 & g_{3N+1} & h_{3N+1} \end{pmatrix}. \tag{36}$$

All solutions of the linear systems $Z\vec{x} = 0$ and $Z\vec{y} = 0$ are given by

$$\vec{x} = u_{1,N}\alpha - u_{2,N}\beta, \qquad \vec{y} = u_{1,N}\beta + u_{2,N}\alpha, \tag{37}$$

where

$$\alpha = (-g_1, \ldots, -g_{3N+1}, 1, 0)^t, \qquad \beta = (h_1, \ldots, h_{3N+1}, 0, -1)^t \tag{38}$$

and $u_{1,N}$, $u_{2,N}$ are free parameters. Moreover, the entries in the last two columns in (36) satisfy

(i) $g_i = h_{i+1}$ *and* $g_{i+1} = -h_i$ *for all* $i = N+2, N+4, \ldots, 3N$,

(ii) $g_i = h_i = 0$ *for all* $1 \leq i \leq \min(N - M_B, \ell)$ *and* $\max(N_B, \ell + N_\Delta - N) + 2 \leq i \leq N + 1$,

(iii) $h_{N_B+1} = 0$, *if* $N_\Delta < N + N_B - \ell$.

Proof: The combined system $Z\vec{x} = 0$, $Z\vec{y} = 0$ is consistent, as mentioned above. Let $r := 3N + 1$. If the first r columns of Z are linearly dependent, there exists a nontrivial solution of $Z\vec{x} = 0$ with $u_{1,N} = u_{2,N} = 0$. Hence, equation (31) has a nontrivial solution (u_1, u_2, v_1) where the degree of u_1 and u_2 is at most $N - 1$. By (ii) and (iii) in Theorem 7, the Laurent polynomial A of degree $2N$ divides the nonzero Laurent polynomial $u_1(z)u_1(1/z) + u_2(z)u_2(1/z)$ of degree less than $2N - 1$. This is a contradiction. Therefore, the first r columns of Z must be linearly independent. This implies that Z has full rank and its reduced row echelon form is given by (36). The vectors α, β in (38) are a basis of the solution set of $Z\vec{x} = 0$. This leads to the form of the solutions for $\vec{x}$ and $\vec{y}$ in (37). Moreover, since the coefficients $u_{1,k}$, $u_{2,k}$ appear in the definition of both vectors $\vec{x}$ and $\vec{y}$, part (i) of the lemma follows by inserting the values $u_{1,N} = 1$ and $u_{2,N} = 0$ in (37).

For parts (ii) and (iii) of the lemma, only the system $Z\vec{x} = 0$ needs to be analyzed. In order to prove part (ii), we define $s = \min(-M_B, \ell - N)$ and inspect the homogeneous equations for the coefficients of z^j in (31), where $-N \le j < s$. Only the product $A(z)v_1(z)$ contributes to these coefficients. This leads to $s + N$ equations of the form

$$-\sum_{k=0}^{j+N} a_{N+k-j} v_{1,k} = 0, \quad -N \le j \le s-1,$$

with unknowns $v_{1,0}, \dots, v_{1,s+N-1}$. These equations are part of the linear system $Z\vec{x} = 0$. Since the matrix of this subsystem is invertible, every solution of $Z\vec{x} = 0$ must have $v_{1,0} = \cdots = v_{1,s+N-1} = 0$. Consequently, both solutions α and β in (38) must have zeros in rows i where $1 \le i \le s + N$, and this gives the first part of (ii). The second part of (ii) follows in an analogous way by inspecting the coefficients of z^j in (31) for $\max(N_B + N, \ell + N_\Delta) + 1 \le j \le 2N$.

Similarly, for part (iii), we inspect the coefficient of z^{N+N_B} in (31). If we have $\ell + N_\Delta < N_B + N$, then the coefficient of z^j, $j = N_B + N$, in (31) vanishes if and only if

$$b_{N_B} u_{1,N} - \sum_{k=N_B}^{N} a_{j-k} v_{1,k} = 0. \tag{39}$$

By part (ii), this can be simplified to $b_{N_B} u_{1,N} - a_N v_{1,N_B} = 0$. Consequently, the solution β in (38) with $u_{1,N} = 0$ and $u_{2,N} = -1$ must have $v_{1,N_B} = h_{N_B+1} = 0$. We have thus shown (iii). $\square$

Parts (ii) and (iii) of the Lemma 8 are essential for the proof of Theorem 4. Before we enter the proof, we specify the following consequence of the lemma.

Lemma 9. *Let the assumptions of Lemma 8 be satisfied. Assume that $B \in L[-N : N-1]$ and $d \in L[0 : 2N - \mu]$, where $\mu \in \{1, 2\}$. Then there exists a nontrivial solution (u_1, u_2, v_1, v_2) of equations (31)–(33), such that*

$$u_1 \in L[0 : N], \quad u_2 \in L[0 : N-1],$$
$$v_1 \in L[0 : N-1], \quad v_2 \in L[0 : N-\mu].$$

Proof: The assumptions on B and d imply that $N_B < N_A = N$ and $\ell + N_\Delta - N \le N - \mu$, where $\mu \in \{1, 2\}$. This gives $\max(N_B, \ell + N_\Delta - N) + 2 \le N + 1$ in part (ii) of Lemma 8. It follows that $g_{N+1} = h_{N+1} = 0$. If $\mu = 2$ and $N_B \le N - 2$, another application of (ii) gives $g_N = h_N = 0$. If $\mu = 2$ and $N_B = N - 1$, however, then Lemma 8(iii) implies that $h_N = 0$. The parameters $u_{1,N} = 1$ and $u_{2,N} = 0$ in (37) define the solution

(u_1, u_2, v_1, v_2) with coefficients $v_{1,N} = -g_{N+1} = 0$ and $v_{2,N} = h_{N+1} = 0$. This is the assertion of Lemma 9 for $\mu = 1$. In the case where $\mu = 2$ we obtain that $v_{2,N-1} = h_N = 0$, which again proves the assertion of the lemma. $\square$

Finally, we are in a position to give the proof of Theorem 4.

Proof of Theorem 4: We have mentioned before that (15) cannot hold if q_1 and q_2 have degree less than N_X. Therefore, let us turn to the existence of Laurent polynomials q_1 and q_2 with the degree constraints deg $q_1 = N_X$ and deg $q_2 \leq N_X - \mu$. We first deal with the case where X and Y have no common symmetric zeros.

Let A, B, C be the Laurent polynomials in (21) and r be chosen as in Lemma 6. Then we obtain the matrix (27), where the Laurent polynomials $\widetilde{A}$, $\widetilde{B}$ have no common zeros and $N_{\tilde{A}} = \max(N_A, N_B, M_B, N_C)$. Recall from (26) and (27) that if N_X is even, we have

$$N_{\tilde{A}} = N_A = N_X/2 > N_C, \quad M_{\tilde{B}} \leq N_X/2, \quad N_{\tilde{B}} \leq N_X/2 - 1.$$

If N_X is odd, we obtain

$$N_{\tilde{A}} = M_{\tilde{B}} = M_B = (N_X + 1)/2, \quad N_A, N_{\tilde{B}}, N_C \leq (N_X - 1)/2.$$

In both cases we conclude that $M_{\tilde{B}} \leq N_{\tilde{A}}$ and $N_{\tilde{B}}, N_C < N_{\tilde{A}}$. Furthermore, we have

$$N_\Delta \leq N_X - \mu \leq 2N_{\tilde{A}} - \ell - \mu,$$

where we let $\ell = 0$, if N_X is even, and $\ell = 1$, if N_X is odd.

Let $N = N_{\tilde{A}}$ as in Lemma 8. The determinant Δ in (18) remains unchanged by the transformation (27). We choose d in (34), such that $d(z)d(1/z) = \Delta(z)$ and $d \in L[\ell : \ell + N_\Delta]$, with $\ell \in \{0, 1\}$ from above. Then the result of Lemma 9 assures, that there exists a nontrivial solution $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)$ of equations (31)–(33), where

$$\begin{aligned} &\tilde{u}_1 \in L[0 : N], \quad \tilde{u}_2 \in L[0 : N-1], \\ &\tilde{v}_1 \in L[0 : N-1], \quad \tilde{v}_2 \in L[0 : N-\mu]. \end{aligned} \tag{40}$$

A nontrivial solution of (31)–(33), where the original Laurent polynomials A and B in (21) are inserted, is obtained by

$$(u_1, u_2, v_1, v_2) = (\tilde{u}_1 - r\tilde{v}_1, \tilde{u}_2 - r\tilde{v}_2, \tilde{v}_1, \tilde{v}_2). \tag{41}$$

If N_X is even, (40) and (41) combined with $N = N_X/2$ lead to

$$\begin{aligned} u_1 \in L[0 : \tfrac{N_X}{2}], \quad u_2 \in L[0 : \tfrac{N_X-2}{2}], \\ v_1 \in L[0 : \tfrac{N_X-2}{2}], \quad v_2 \in L[0 : \tfrac{N_X-2\mu}{2}]. \end{aligned}$$

The corresponding Laurent polynomials q_1 and q_2 in (22) satisfy

$$q_1 \in L[2\ell_1 - \nu : 2\ell_1 - \nu + N_X], \quad q_2 \in L[2\ell_2 - \nu : 2\ell_2 - \nu + N_X - \mu].$$

This shows the existence result of the theorem, if N_X is even. If N_X is odd, the choice of $\ell = 1$ in (34) has the effect that the homogeneous equations

$$\tilde{b}_{-N}\tilde{u}_{i,0} - \tilde{a}_{-N}\tilde{v}_{i,0} = 0, \qquad i = 1, 2, \tag{42}$$

follow from (31) (for $i = 1$) and (32) (for $i = 2$). Here, $\tilde{b}_{-N}$, $\tilde{a}_{-N}$ denote the coefficients of the monomial z^{-N} in the Laurent polynomials $\widetilde{B}$ and $\widetilde{A}$. The definition of $\widetilde{A}$ and $\widetilde{B}$ in (27) combined with $N = M_B > N_A, N_B, N_C$ give $\tilde{b}_{-N} = b_{-N}$ and $\tilde{a}_{-N} = rb_{-N}$. Therefore, the solutions of (42) satisfy $\tilde{u}_{i,0} - r\tilde{v}_{i,0} = 0$, $i = 1, 2$. This, together with (40), (41), leads to

$$\begin{aligned} &u_1 \in L[1 : \tfrac{N_X+1}{2}], \quad u_2 \in L[1 : \tfrac{N_X-1}{2}], \\ &v_1 \in L[0 : \tfrac{N_X-1}{2}], \quad v_2 \in L[0 : \tfrac{N_X+1-2\mu}{2}]. \end{aligned}$$

Inserting these polynomials into (22) gives

$$\begin{aligned} &q_1 \in L[2\ell_1 - \nu + 1 : 2\ell_1 - \nu + N_X + 1], \\ &q_2 \in L[2\ell_2 - \nu + 1 : 2\ell_2 - \nu + N_X + 1 - \mu]. \end{aligned}$$

This completes the proof of the theorem under the additional constraint that X and Y have no common symmetric zeros.

If X and Y have a common symmetric zero $z_0 \in \mathbb{C} \setminus \{0\}$, then $1/z_0$ must also be a common symmetric zero, since $X(z) = X(1/z)$ and $Y(-z) = Y(1/z)$ hold by (17). Moreover, $\overline{z_0}$ and $1/\overline{z_0}$ are common symmetric zeros as well, since X and Y have real coefficients. Let z_k, $1 \le k \le 2\kappa$, denote all common symmetric zeros of X and Y. Since the matrix in (15) is positive semi-definite on $\mathbb{T}$, each zero $z_k \in \mathbb{T}$ must have even multiplicity. We can therefore order the zeros such that $z_k z_{k+\kappa} = 1$, $1 \le k \le \kappa$, and $\prod_{k=1}^{\kappa}(z - z_k)$ is a polynomial with real coefficients. The division of X and Y by all factors $(z - z_k)(1/z - z_k)$, $1 \le k \le \kappa$, leads to new Laurent polynomials $\widetilde{X}$, $\widetilde{Y}$ that have real coefficients and the same form (17) as X and Y, where $N_{\tilde{X}} = N_X - \kappa$. The parameter N_Δ of the determinant is reduced by 2κ. Therefore, the parameter μ in the theorem remains 2 or increases from 1 to 2. An application of the result proved so far leads to Laurent polynomials $\tilde{q}_i$, $i = 1, 2$, that satisfy (15) (with $\widetilde{X}$ and $\widetilde{Y}$ instead of X and Y) and have $\deg \tilde{q}_1 = N_{\tilde{X}}$, $\deg \tilde{q}_2 =\le N_{\tilde{X}} - 2$. Multiplication by all factors $(z - z_k)$, $1 \le k \le \kappa$, gives Laurent polynomials q_1, q_2 with real coefficients that satisfy the assertion of the theorem. $\square$

We summarize the computational method for finding the minimum degree solutions q_1, q_2 of (15), whose degree does not exceed N_X, in the following algorithm.

Algorithm. *Assume that X and Y are given as in (16)–(17), and that X and Y have no common symmetric roots. Then minimum degree solutions of (15) whose degree does not exceed N_X are found by the following procedure:*

1) *Let N denote the largest integer less than or equal to $(N_X+1)/2$. Choose ν as in (20) and compute the Laurent polynomials A, B, C in (21).*
2) *Choose $r \in \mathbb{R}$ such that the assumptions of Lemma 8 for the modified Laurent polynomials $\widetilde{A}$, $\widetilde{B}$, C are satisfied. Almost every $r \in \mathbb{R}$ is suitable.*
3) *For each choice of d in (34), where $d \in L[0:2N]$ and $d(z)d(1/z) = \Delta(z)$:*
   - 3a) *Compute the matrix Z in the matrix representation of equation (31), using the vector $\vec{x}$ in (35) as unknowns. (Columns of Z repeat the coefficient sequences of $-\widetilde{A}$, $\widetilde{B}$, and the reverse coefficient sequence of $-d$, each padded with zeros on top and/or bottom.)*
   - 3b) *Find the reduced row echelon form of Z.*
   - 3c) *Among all nontrivial choices of parameters $\tilde{u}_{1,N}$, $\tilde{u}_{2,N}$ for the computation of $\vec{x}$ and $\vec{y}$ in (37), find those for which the pair*

$$q_1(z) = z^{-\nu}(\tilde{u}_1(z^2) + (z-r)\tilde{v}_1(z^2)),$$
$$q_2(z) = z^{-\nu}(\tilde{u}_2(z^2) + (z-r)\tilde{v}_2(z^2))$$

   has minimum degree.
4) *Multiply the minimum degree solution(s) (q_1, q_2) found in step 3 by a positive constant so as to satisfy $q_1^2(1) + q_2^2(1) = X(1)$.*

The result is a pair (q_1, q_2) with $\deg q_1 = N_X$, $\deg q_2 \le N_X - \mu$, *where* $\mu = \min(2, N_X - N_\Delta)$.

If X and Y have common symmetric zeros, the conclusions (ii) and (iii) of Theorem 7 may fail. Part (i) of that theorem remains valid, however. This implies the necessity of equations (31)–(33). N is chosen to be the maximum of N_A, M_B, N_B, N_C. Therefore, in the absence of conclusion (ii) of the theorem, the combined system $(Z\vec{x} = 0,\ Z\vec{y} = 0)$ should be solved. Nontrivial solutions exist by Theorem 4. They must be cross-checked, however, with (28), since the conclusion (iii) of Theorem 7 may fail as well. This search would produce the minimum degree solutions (q_1, q_2) of (15). On the other hand, it may be much simpler to work with the reduced Laurent polynomials $\widetilde{X}$, $\widetilde{Y}$, which are X and Y, respectively, divided by linear factors that contain symmetric zeros as in parts of the proof of Theorem 4. This will produce solutions (q_1, q_2) of (15), where q_1 and q_2 have as common zeros half of the common symmetric zeros of X and Y. (The other half appears in $q_1(1/z)$ and $q_2(1/z)$ in (15).) Their degrees satisfy the inequalities in Theorem 4. It is not clear, however, that the minimum degree solution (q_1, q_2) of (15) can be obtained in this way.

The following example serves as an illustration of our algorithm.

Example 1. Find Laurent polynomials (q_1, q_2) of minimum degree such that (15) holds, where

$$X(z)=5(z^6+z^{-6})+14(z^5+z^{-5})+26(z^4+z^{-4})+28(z^3+z^{-3})+49(z^2+z^{-2})+74(z+z^{-1})+122,$$

$$Y(z)=5(z^6+z^{-6})+6(z^5+z^{-5})+10(z^4+z^{-4})+14(z^3+z^{-3})+45(z^2+z^{-2})+16(z+z^{-1})+40.$$

Neither X nor Y have symmetric zeros in $\mathbb{C}$. Therefore, the algorithm can be started with $N_X = 6$, $N = 3$ and $\nu = 0$ in (20). The polyphase decomposition reveals

$$\begin{aligned}
A(z) &= 5(z^3 + z^{-3}) + 18(z^2 + z^{-2}) + 47(z + z^{-1}) + 81, \\
C(z) &= 8(z^2 + z^{-2}) + 2(z + z^{-1}) + 41, \\
B(z) &= 4z^{-3} + 7z^{-2} + 29z^{-1} + 45 + 21z + 10z^2
\end{aligned}$$

and

$$\Delta(z) = 325 - 150(z + z^{-1}), \qquad N_\Delta = 1.$$

Hence, we obtain $\mu = 2$ in Theorem 4 and know that solutions (q_1, q_2) exist with $\deg q_1 = 6$ and $\deg q_2 \le 4$. Step 2 of the algorithm can be skipped ($r = 0$ is suitable in Lemma 6). Since Δ has two roots $z_1 = 3/2$, $z_2 = 2/3$, there are 12 choices for $d \in L[0 : 6]$ in step 3 of the algorithm. They are given by

$$d(z) = z^\ell(15z - 10), \quad d(z) = z^\ell(10z - 15), \quad 0 \le \ell \le 5.$$

We demonstrate step 3 of the algorithm for $d(z) = 15z - 10$. Equation (31) leads to a linear system of 10 equations for 12 unknowns

$$\vec{x} = (v_{1,0}, \dots, v_{1,3}, u_{1,0}, u_{2,0}, \dots, u_{1,3}, u_{2,3})^t.$$

The last two columns of the reduced row-echelon form of Z are

$$\begin{pmatrix} -4 & 1 & -2 & 0 & -5 & -5 & -3 & -4 & -1 & -2 \\ 2 & 4 & 0 & 0 & 5 & -5 & 4 & 3 & 2 & -1 \end{pmatrix}^t.$$

The patterns described in Lemma 8 can be recognized. The choice $u_{1,N} = 1$, $u_{2,N} = 0$ leads to minimum degree solutions

$$\begin{aligned}
q_1(z) &= (5 + 3z^2 + z^4 + z^6) + z(4 - z^2 + 2z^4), \\
q_2(z) &= (5 + 4z^2 + 2z^4) + z(2 + 4z^2).
\end{aligned}$$

Since $q_1^2(1) + q_2^2(1) = 514 = X(1)$, q_1 and q_2 possess the correct normalization. We verified that the other cases do not lead to solutions with smaller degree.

Acknowledgment. This research was supported by NSF Grants CCR-9988289 and CCR-0098331 and ARO Grant DAAD19-00-1-0512.

References

1. Chui, C. K., *An Introduction to Wavelets*, Academic Press, Boston, 1992.
2. Chui, C. K., W. He and J. Stöckler, Compactly supported tight and sibling frames with maximal vanishing moments, Appl. Comput. Harmonic Anal., to appear.
3. Chui, C. K., W. He, J. Stöckler and Q. Sun, Compactly supported tight affine frames with integer dilations and maximum vanishing moments, J. Fourier Anal. and Appl., to appear.
4. Daubechies, I., B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmonic Anal., to appear.
5. Han, B., On dual wavelet tight frames, Appl. Comput. Harmon. Anal. **4** (1997), 380–413.
6. Hernández, E., and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, 1996.
7. Jia, R.-Q., and Z. Shen, Multiresolution and wavelets, Proc. Edinburgh Math. Soc. **37** (1994), 271–300.
8. Jia, R.-Q., and J. Z. Wang, Stability and linear independence associated with wavelet decompositions, Proc. Amer. Math. Soc. **117** (1993), 1115–1124.
9. Lawton, W., S. L. Lee and Z. Shen, Stability and orthonormality of multivariable refinable functions, SIAM J. Math. Anal. **28** (1997), 999–1014.
10. Pólya, G., and G. Szegö, *Problems and Theorems in Analysis II*, (reprint ed.), Springer, Heidelburg, 1998.
11. Rheinboldt, W. C., and J. S. Vandergraft, A simple approach to the Perron-Frobenius theory for positive operators on general partially-order finite-dimensional linear spaces, Math. Comp. **27** (1973), 139–145.
12. Ron, A., and Z. W. Shen, Affine systems in $L_2(R^d)$: the analysis of the analysis operator, J. Funct. Anal. **148** (1997), 408–447.

Charles K. Chui and Wenjie He
University of Missouri–St. Louis
Department of Mathematics
St.Louis, MO 63121-4499

Joachim Stöckler
Universität Dortmund
Fachbereich Mathematik
44221 Dortmund, Germany

`cchui@stat.stanford.edu` `he@neptune.cs.umsl.edu`
`joachim.stoeckler@math.uni-dortmund.de`