[9] B. Kisačanin and D. Schonfeld, "A fast threshold linear convolution representation of morphological operations," *IEEE Trans. Image Processing*, vol. 3, pp. 455–457, July 1994.
[10] H. V. Poor, *An Introduction to Signal Detection and Estimation*, New York: Springer-Verlag, 1994.
[11] H. J. A. M. Heijmans, *Morphological Image Operators*. San Diego, CA: Academic, 1994.
[12] D. Schonfeld, "Fast parallel nonlinear filtering based on the FFT," in *Proc. Int. Conf. Digital Signal Processing*, vol. 1, Limassol, Cyprus, 1995, pp. 338–341.

Examples of Bivariate Nonseparable Compactly Supported Orthonormal Continuous Wavelets

Wenjie He and Ming-Jun Lai

***Abstract*—We give many examples of bivariate nonseparable compactly supported orthonormal wavelets whose scaling functions are supported over [0, 3] × [0, 3]. The Hölder continuity properties of these wavelets are studied.**

***Index Terms*—Compact support, continuous, nonseparable, orthonormal, wavelet.**

I. Introduction

Univariate wavelets have found successful applications in signal processing since wavelet expansions are more appropriate than conventional Fourier series to represent the abrupt changes in nonstationary signals. To apply wavelet methods to digital image processing, we have to construct vibariate wavelets. The most commonly used method is the tensor product of univariate wavelets. This construction leads to a separable wavelet which has a disadvantage of giving a particular importance to the horizontal and vertical directions. Much effort (cf., e.g., [1]–[3]) has been spent on constructing nonseparable bivariate wavelets. In this paper, we construct vibariate nonseparable compactly supported orthonormal wavelets based on the commonly used uniform dilation matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Let

$$m_o(\omega) := m_0(\omega_1, \omega_2) = \sum_{0 \le j \le p, 0 \le k \le q} c_{j,k} \exp(i(j\omega_1 + k\omega_2))$$

be a trigonometric polynomial. We will construct m_0 which satisfies the following requirements: 1°: $m_0(0,0) = 1$; 2°: $\Sigma_{j=0}^3 \; |m_0(\omega + \pi_j)|^2 = 1$ with $\pi_0 = (0,0)$, $\pi_1 = (\pi, 0)$ $\pi_2 = (0, \pi)$, and $\pi_3 = (\pi, \pi)$. Let $\hat{\phi}(\omega) = \Pi_{k=1}^{\infty} \; m_0(\omega/2^k)$ be generated by m_0. Then 1° implies the convergence of this infinite product and hence $\hat{\phi}$ is a well-defined continuous function. 2° implies $\hat{\phi} \in L_2(\mathbf{R}^2)$. Thus, $\phi \in L_2(\mathbf{R}^2)$ by Plancheral's Theorem. For a fixed ordering which maps bi-integers (0, 0) ≤ $(j, k) \le (p, q)$ into positive integers $\{1, 2, \cdots, N\}$ with $N = (p+1)(q+1)$, let A be a matrix of size $N \times N$ with entries

$$A_{k_1, k_2; \ell_1, \ell_2} = 4 \sum_{j_1, j_2} c_{j_1, j_2} \overline{c_{(j_1, j_2) + (k_1, k_2) - 2(\ell_1, \ell_2)}}$$

for (0, 0) ≤ $(k_1, k_2, (\ell_1, \ell_2) \le (p, q)$. In order to make $\{\phi(x - k_1, y - k_2), (k_1, k_2) \in Z^2\}$ to be an orthonormal set, we need to have the bivariate generalizationof the Lawton condition 3° (cf. [4]). One is a nondegenerate eigenvalue of A. We then further study the coefficients of m_0 such that 4°: $\phi \in C^{\gamma}(\mathbf{R}^2)$ with $\gamma \ge 0$. After these preparations, we shall construct $m_\nu, \nu = 1, 2, 3$, such that 5°: $\sum_{j=0}^3 \; m_\mu(\omega + \pi_j)\overline{m_\nu(\omega + \pi_j)} = \delta_{\mu,\nu}$, $\mu, \nu = 0, 1, 2, 3$. To make m_0 to be a low-pass filter, we require that m_0 have a factor $(1 + e^{i\omega})\,(1 + e^{i\omega_2})$. That is, 6°: $m_0(\pi, \omega_2) = 0 = m_0(\omega_1, \pi)$ for all $(\omega_1, \omega_2) \in [-\pi, \pi]$. For $p = q = 3$, we are able to give a complete solution set of all m_0 satisfying 1°, 2°, and 6°. We identify many families of solutions which further satisfy 3° and 4°. For example, a tensor product of Daubechies' scaling function ${}_2\phi$ is included. It is known that ${}_2\phi(x_1)_2\phi(x_2) \in C^{\alpha}(\mathbf{R}^2)$ with $\alpha \ge 0.5$ [5]. We can expect other solutions to have certain Hölder's exponents. We study the regularity of those filters. Finally, we construct m_ν to satisfy 5° for any given m_0 satisfying 1° and 2°. In Section III, we present some numerical experiments using our nonseparable wavelets.

II. Construction of Scaling Functions and Wavelets

Rewrite $m_0(\omega_1, \omega_2)$ as $m(x, y) = \Sigma_{0 \le j \le p, 0 \le k \le q} \; c_{j,k} x^j y_k$ with $x = e^{i\omega_1}$ and $y = e^{i\omega_2}$. Also write $m(x, y)$ in its polyphase form: $m(x,y) = f_0(x^2, y^2) + s f_1(x^2, y^2) + y f_2(x^2, y^2) + xy f_3(x^2, y^2)$. It is well-known that a polynomial m satisfying 2° is equivalent to

$$|f_0|^2 + |f_1|^2 + |f_2|^2 + |f_3|^2 = \tfrac{1}{4}. \tag{1}$$

From now on, we only consider $p = q = 3$. Thus, we write $f_\nu(x, y) = a_\nu + v_\nu x + c_\nu y + d_\nu xy$, ν = 0, 1, 2, 3. We now present one of the main results in this paper.

Theorem 2.1.: Let

$$m(x,y) = \frac{(1+x)(1+y)}{16}(a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2 + a_{02}y^2) \tag{2}$$

with

$$\begin{aligned}
a_{00} &= 1 + \sqrt{2}(\cos\alpha + \cos\beta) + 2\cos\theta\cos\xi \\
a_{10} &= \sqrt{2}(\sin\alpha - \cos\alpha) - 2\cos\theta\cos\xi + 2\cos\theta\sin\xi \\
a_{01} &= \sqrt{2}(\sin\beta - \cos\beta) - 2\cos\theta\cos\xi + 2\sin\theta\cos\eta \\
a_{11} &= 2(\cos\theta\cos\xi + \sin\theta\sin\eta - \cos\theta\sin\xi - \sin\theta\cos\eta) \\
a_{20} &= 1 + \sqrt{2}(\cos\beta - \sin\alpha) - 2\cos\theta\sin\xi \\
a_{02} &= 1 + \sqrt{2}(\cos\alpha - \sin\beta) - 2\sin\theta\cos\eta \\
a_{21} &= \sqrt{2}(\sin\beta - \cos\beta) - 2\sin\theta\sin\eta + 2\cos\theta\sin\xi \\
a_{12} &= \sqrt{2}(\sin\alpha - \cos\alpha) - 2\sin\theta\sin\eta + 2\sin\theta\cos\eta \\
a_{22} &= 1 - \sqrt{2}(\sin\alpha + \sin\beta) + 2\sin\theta\sin\eta.
\end{aligned} \tag{3}$$

Then, $m(x, y)$ satisfies 2° if $\alpha, \beta, \theta, \xi, \eta$ satisfy the following:

$$\cos\theta\cos\xi + \cos\theta\sin\xi + \sin\theta\cos\eta + \sin\theta\sin\eta = 2\sin\left(\alpha + \frac{\pi}{4}\right)\sin\left(\beta + \frac{\pi}{4}\right). \tag{4}$$

Proof: It is straightforward to verify that f_0, f_1, f_2 and f_3 satisfy (1) if and only if

$$\sum_{\nu=0}^{3} (a_\nu b_\nu + c_\nu d_\nu) = 0, \quad \sum_{\nu=0}^{3} (a_\nu c_\nu + b_\nu d_\nu) = 0,$$
$$\sum_{\nu=0}^{3} a_\nu d_\nu = 0, \quad \sum_{\nu=0}^{3} b_\nu c_\nu = 0 \tag{5}$$

Manuscript received July 24, 1997; revised August 4, 1999. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Kannan Ramchandran.

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Publisher Item Identifier S 1057-7149(00)01153-2.

and

$$\sum_{\nu=0}^{3}(a_\nu^2+b_\nu^2+c_\nu^2+d_\nu^2)=\tfrac{1}{4}. \tag{6}$$

It follows from (5) and (6) that

$$\sum_{\nu=0}^{3}(a_\nu-b_\nu)^2+(c_\nu-d_\nu)^2=\tfrac{1}{4} \quad \text{and}$$
$$\sum_{\nu=0}^{3}(a_\nu-b_\nu)(c_\nu-d_\nu)=0.$$

Then we have

$$\sum_{\nu=-}^{3}(a_\nu-b_\nu+c_\nu-d_\nu)^2=\tfrac{1}{4}.$$

By 1° and 6°, it follows that

$$\sum_{\nu=0}^{3}(2(a_\nu+c_\nu)-\tfrac{1}{4})^2=\tfrac{1}{4}.$$

Now 6° implies that $a_0+c_0=a_2+c_2$ and $a_1+c_1=a_3+c_3$. It follows that

$$\begin{aligned}a_0+c_0&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\cos\alpha=a_2+c_2 \quad \text{and}\\ a_1+c_1&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\sin\alpha=a_3+a_3.\end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned}a_0+b_0&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\cos\beta=a_1+b_1 \quad \text{and}\\ a_2+b_2&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\sin\beta=a_3+b_3.\end{aligned} \tag{8}$$

In other words, we have

$$\begin{aligned}b_i&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\cos\beta-a_i, i=0,1,\\ b_i&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\sin\beta-a_i, i=2,3,\\ c_i&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\cos\alpha=a_i, i=0,2,\\ c_i&=\frac{1}{8}+\frac{1}{4\sqrt{2}}\sin\alpha-a_i, i=1,3,\\ d_0&=-\frac{1}{4\sqrt{2}}(\cos\alpha+\cos\beta)+a_0,\\ d_1&=-\frac{1}{4\sqrt{2}}(\sin\alpha+\cos\beta)+a_1,\\ d_2&=-\frac{1}{4\sqrt{2}}(\cos\alpha+\sin\beta)+a_2,\\ d_3&=-\frac{1}{4\sqrt{2}}(\sin\alpha+\sin\beta)+a_3.\end{aligned} \tag{9}$$

We now find the relations among the a_ν's. By (5), specifically, $0=\sigma_{\nu=0}^{3}\ a_\nu d_\nu$, we have

$$\begin{aligned}&\left(a_0-\frac{1}{8\sqrt{2}}(\cos\alpha+\cos\beta)\right)^2+\left(a_1-\frac{1}{7\sqrt{2}}(\sin\alpha+\cos\beta)\right)^2\\ &\quad+\left(a_2-\frac{1}{8\sqrt{2}}(\cos\alpha+\sin\beta)\right)^2\\ &\quad+\left(a_3-\frac{1}{8\sqrt{2}}(\sin\alpha+\sin\beta)\right)^2\\ &=\frac{1}{32}\left(1+\sin\left(\alpha+\frac{\pi}{4}\right)\sin\left(\beta+\frac{\pi}{4}\right)\right).\end{aligned} \tag{10}$$

By (5), specifically, $\Sigma_{\nu=0}^{3}\ b_\nu c_\nu=0$ and $\Sigma_{\nu=0}^{2}\ a_\nu d_\nu=0$, we have

$$\begin{aligned}&a_0-\frac{1}{8\sqrt{2}}(\cos\alpha+\cos\beta)+a_1-\frac{1}{8\sqrt{2}}(\sin\alpha+\cos\beta)\\ &\quad+a_2-\frac{1}{8\sqrt{2}}(\cos\alpha+\sin\beta)\\ &\quad+a_3-\frac{1}{8\sqrt{2}}(\sin\alpha+\sin\beta)\\ &=\frac{1}{4}\left(1+\sin\left(\alpha+\frac{\pi}{4}\right)\sin\left(\beta+\frac{\pi}{4}\right)\right).\end{aligned} \tag{11}$$

Let

$$\begin{aligned}\tilde{a}_0&=a_0-\frac{1}{8\sqrt{2}}(\cos\alpha+\cos\beta),\\ \tilde{a}_1&=a_1-\frac{1}{8\sqrt{2}}(\sin\alpha+\cos\beta),\\ \tilde{a}_2&=a_2-\frac{1}{8\sqrt{2}}(\cos\alpha+\sin\beta),\\ \tilde{a}_3&=a_3-\frac{1}{8\sqrt{2}}(\sin\alpha+\sin\beta).\end{aligned}$$

Equations (10) and (11) become

$$(\tilde{a}_0-\tfrac{1}{16})^2+(\tilde{a}_1-\tfrac{1}{16})^2+(\tilde{a}_2-\tfrac{1}{16})^2+(\tilde{a}_3-\tfrac{1}{16})^2=(\tfrac{1}{8})^2.$$

It follows that

$$\begin{aligned}a_0&=\frac{1}{16}+\frac{1}{8\sqrt{2}}(\cos\alpha+\cos\beta)+\frac{1}{8}\cos\theta\cos\xi,\\ a_1&=\frac{1}{16}+\frac{1}{8\sqrt{2}}(\sin\alpha+\cos\beta)+\frac{1}{8}\cos\theta\sin\xi,\\ a_2&=\frac{1}{16}+\frac{1}{8\sqrt{2}}(\cos\alpha+\sin\beta)+\frac{1}{8}\sin\theta\cos\eta,\\ a_3&=\frac{1}{16}+\frac{1}{8\sqrt{2}}(\sin\alpha+sin\beta)+\frac{1}{8}\sin\theta\sin\eta.\end{aligned} \tag{12}$$

By (11) and (12), $\alpha,\beta,\theta,\xi,\eta$ must satisfy

$$\begin{aligned}&\frac{1}{4}+\frac{1}{8}(\cos\theta\cos\xi+\cos\theta\sin\xi+\sin\theta\cos\eta+\sin\theta\sin\eta)\\ &\quad=\frac{1}{4}+\frac{1}{4}\sin\left(\alpha+\frac{\pi}{4}\right)\sin\left(\beta+\frac{\pi}{4}\right).\end{aligned}$$

After simplified, the above equation is (4). The above derivations show that any solution $m(x,y)$ satisfying 1°, 2°, and 6° must be in the form (2) and (3) with $\alpha,\beta,\gamma,\xi,\eta$ satisfying (4).

On the other hand, any solution m in the form (2) and (3) with $\alpha,\beta,\gamma,\xi,\eta$ satisfying (4) will satisfy (10) and (11). Equations (10) and (11) are equivalent to $\Sigma_{\nu=0}^{3}\ a_\nu d_\nu=0$ and $\Sigma_{\nu=0}^{3}\ b_\nu c_\nu=0$. By the expressions in (3), we have (7)–(9). These equations imply that

$$\begin{aligned}\sum_{\nu=0}^{3}(a_\nu-b_\nu+c_\nu-d_\nu)^2&=\tfrac{1}{4}\\ \sum_{\nu=0}^{3}(a_\nu-b_\nu-c_\nu+d_\nu)^2&=\tfrac{1}{4}\\ \sum_{\nu=0}^{3}(a_\nu+b_\nu+c_\nu+d_\nu)^2&=\tfrac{1}{4}.\end{aligned}$$

The above three equations are equivalent to the first two equations in (5) and (6). ■

Let us present several families of filters $m(x,y)$ and use 3° to check their orthonormality. We will use the following theorem to check their regularity. Recall ϕ is the scaling function generated by filters $m(e^{i\omega_1},e^{i\omega_2})$. To check if $\phi\in C^\gamma(\mathbf{R}^2)$, we study the finiteness of $\int_{\mathbf{R}^2}\ |\hat{\phi}(\omega)|(1+|\omega|^\gamma)\ d\omega$. Writing

$$p_0(\omega_1,\omega_2)=\sum_{0\le j,k\le 2}\ a_{j,k}e^{i(j\omega_1+k\omega_2)}$$

we define an operator P acting on a trigonometric polynomial space

$$E := \left\{ \sum_{-1\le j,k\le 1}^{0\le j,k\le 2} c_{jk} e^{i(j\omega_1 + k\omega_2)} \colon c_{j,k} \in \mathbf{R} \right\}$$

by

$$(Pf)(\omega) = \sum_{i=0}^{3} \left| p_o\left(\frac{\omega}{2} + \pi_i\right)\right|^2 f\left(\frac{\omega}{2} + \pi_i\right), \quad \forall f \in E. \tag{13}$$

Theorem 2.2: Let $f_0 = 1$ and λ be the spectral radius of the operator P so that

$$\int_{[-\pi,\pi]^2} (P^n f_0)(\omega)\, d\omega \le C(\lambda + \delta)^n \tag{14}$$

for a sufficiently small δ. If $\lambda < 2$, then ϕ generated by $m(e^{i\omega_1}, e^{i\omega_2})$ is continuous. Further, $\phi \in C^\gamma(\mathbf{R}^2)$ for $\gamma < |, (1/2)\log_2(2/\lambda)$. See [6] for a proof. We have the following families of nonseparable filters.

Example 2.1: We first look for separable filters. If we set $\xi = \eta$ in (4), the (4) becomes $(\cos\theta + \sin\theta)(\cos\xi + \sin\xi) = (\cos\alpha + \sin\alpha)(\cos\beta + \sin\beta)$. If we further choose $\theta = \beta$, then $\xi = \alpha$ and $m(x, y)$ in (2) may be simplified into $m(x, y) = M(x, \beta)M(y, \alpha)$ with

$$\begin{aligned} M(x,\alpha) = \frac{1+x}{4}&(1 + \sqrt{2}\cos\alpha \\ &+ \sqrt{2}(\sin\alpha - \cos\alpha)x + (1 - \sqrt{2}\sin\alpha)x^2). \end{aligned} \tag{15}$$

For

$$\begin{aligned} \alpha = \beta &= \frac{5\pi}{12}, \\ M(e^{i\omega}, 5\pi/12) &= \frac{1}{2}\left[\frac{1+\sqrt{3}}{4} + \frac{3+\sqrt{3}}{4}e^{i\omega} \right. \\ &\quad \left. + \frac{3-\sqrt{3}}{4}e^{i2\omega} + \frac{1-\sqrt{3}}{4}e^{i3\omega}\right] \end{aligned}$$

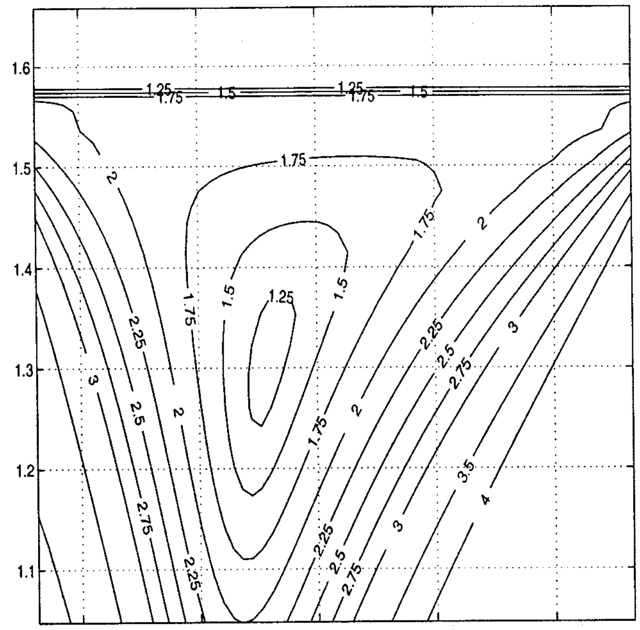
is the filter associated with Daubechies' scaling function ${}_2\phi$ [5]. Hence, for $\theta = \beta$ and $\xi = \eta = \alpha$, (4) is satisfied and $m(x, y)$ given in (2) is a separable filter.

It is easy to write down the Lawton matrix A associated with $M(e^{i\omega}, \alpha)$ (see [6]). For any $\alpha \in [0, 2\pi]$ except for $\alpha = -\pi/4$, the eigenvalue 1 of A is not degenerate. Hence, $m_0(\omega_1, \omega_2) = M(e^{i\omega_1}, \alpha)M(e^{i\omega_2}, \beta)$ generates an orthonormal scaling function ϕ in $L_2(\mathbf{R}^2)$ if $\alpha \ne -(\pi/4)$ and $\beta \ne -(\pi/4)$. In [7], Colella and Heil gave a detail study of the continuity of the orthonormal scaling functions supported on [0, 3] in the univariate setting. We may use the method discussed here to study the continuity using $M(x, \alpha)$ from (15). We leave the detail to [6].

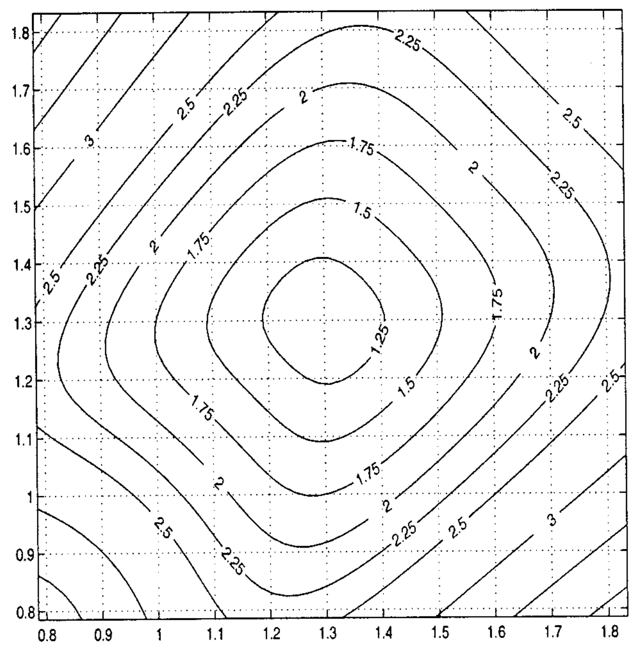
Example 2.2: Let us look for linear phase filters. That is, we need to have $a_{00} = a_{22}, a_{02} = a_{20}, a_{12} = a_{10}, a_{01} = a_{21}$. Solving these four equations together with (4), we obtain eight filters satisfying 1°, 2°, and 6° with a fixed length $p = q = 3$ up to a shift. We refer to [6] for details. Using 3° to check these eight filters, we find that the first four filters generate an orthonormal scaling function while the last four filters do not. All ϕ generated by these eight filters are only in $L_2(\mathbf{R}^2)$, none are continuous.

Example 2.3: Let us consider the filter $m(x, y)$ symmetric with respect to the line $x = y$. Then $a_{12} = a_{21}, a_{02} = a_{20}, a_{01} = a_{10}$. These imply $\alpha = \beta$ and $\sin\theta\cos\eta = \cos\theta\sin\xi$. Thus by (4), these $\alpha, \beta, \theta, \xi, \eta$ must satisfy

$$\begin{aligned} &\cos\theta\cos\xi + 2\cos\theta\sin\xi + \sin\theta\sin\eta \\ &\quad = 2\sin^2\left(\alpha + \frac{\pi}{4}\right) \text{ and } \sin\theta\cos\eta = \cos\theta\sin\xi. \end{aligned} \tag{16}$$



(a)

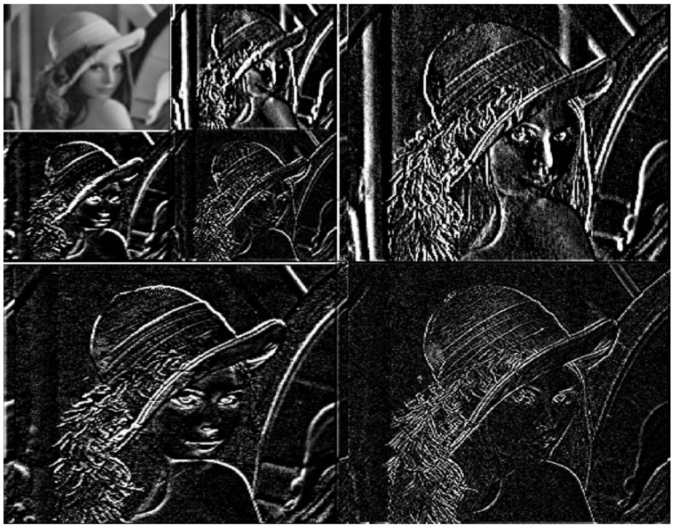


(b)

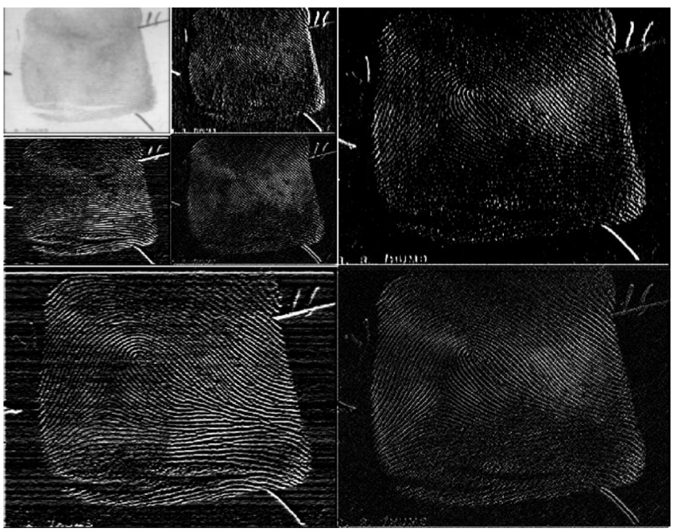
Fig. 1. Contour of spectrum λ based on (a) $(\xi, \eta) \in [\pi/10, 9\pi/10] \times [\pi/3, 19\pi/36]$ and (b) $(\theta, \xi) \in [\pi/4, 7\pi/12]^2$.

There are infinite possible ξ and η for which we can find θ and α satisfying the two equations in (16), including $(\xi, \eta) \in [\pi/10, 9\pi/10] \times [\pi/3, 19\pi/36]$. Next let us give two examples which have rational coefficients:

$$\begin{aligned} m(x,y) &= \frac{(1+x)(1+y)}{100}(11 + 6x - 2x^2 + 6y \\ &\quad + 13xy - 4x^2y - 2y^2 - 4xy^2 + x^2y^2), \\ m(x,y) &= \frac{(1+x)(1+y)}{3468}(544 + 120x - 52x^2 \\ &\quad + 120y + 416xy - 128x^2y - 52y^2 \\ &\quad - 128xy^2 + 27x^2y^2). \end{aligned}$$



(a)



(b)

Fig. 2. Decomposition by a nonseparable filter.

Using 3°, all the filters above generate an orthonormal scaling functions. We now check the regularity of the scaling functions ϕ. Let $f_0(\omega_1, \omega_2) = 1$. Then

$$Pf_0(2\omega_1, 2\omega_2) = \sum_{\nu=0}^{4} q_{0,\nu} f_\nu(2\omega_1, 2\omega_2)$$

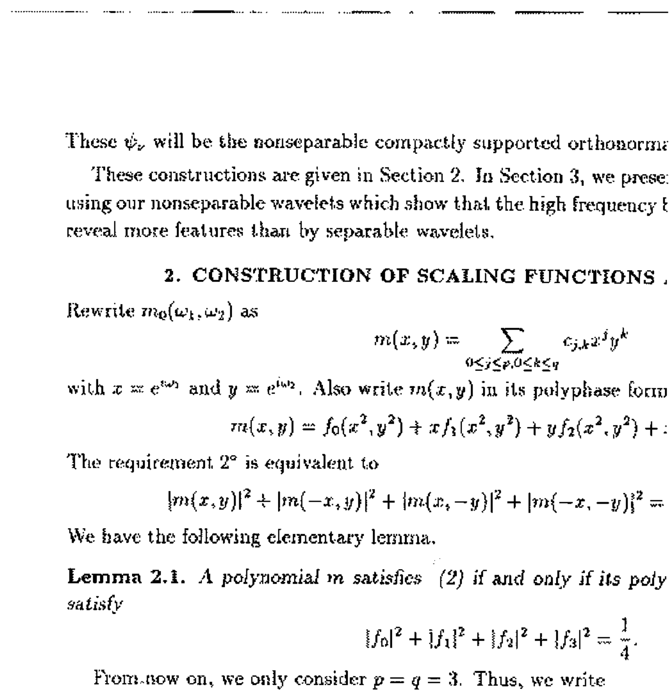
with

$$\begin{aligned} f_1(\omega_1, \omega_2) &= e^{i\omega_1} + e^{-i\omega_1}, \\ f_3(\omega_1, \omega_2) &= e^{i(\omega_1+\omega_2)} + e^{-i(\omega_1+\omega_2)}, \\ f_2(\omega_1, \omega_2) &= e^{i\omega_2} + e^{-i\omega_2}, \\ f_4(\omega_1, \omega_2) &= e^{i(\omega_1-\omega_2)} + e^{-i(\omega_1-\omega_2)}. \end{aligned}$$

Then

$$Pf_\nu(2\omega_1, 2\omega_2) = \sum_{\nu=0}^{4} q_{\nu,\mu} f_\mu(2\omega_1, 2\omega_2).$$

Under the basis $\{f_0, f_1, f_2, f_3, f_4\}$, the matrix for the P is $[q_{\mu,\nu}]_{0\le\mu,\nu\le 4}$. We plot the largest eigenvalue as a function of $(\xi, \eta) \in [\pi/10, 9\pi/10] \times [\pi/3, 19\pi/36]$ on the left graph of Fig. 1. Theorem 2.2 implies all scaling functions within the contour line $\lambda = 2$ as shown in Fig. 1(a) are continuous.



These ψ_ν will be the nonseparable compactly supported orthonorma

These constructions are given in Section 2. In Section 3, we prese
using our nonseparable wavelets which show that the high frequency b
reveal more features than by separable wavelets.

2. CONSTRUCTION OF SCALING FUNCTIONS

Rewrite $m_0(\omega_1, \omega_2)$ as

$$m(x,y) = \sum_{0\le j\le p, 0\le k\le q} c_{j,k} x^j y^k$$

with $x = e^{i\omega_1}$ and $y = e^{i\omega_2}$. Also write $m(x,y)$ in its polyphase form

$$m(x,y) = f_0(x^2,y^2) + x f_1(x^2,y^2) + y f_2(x^2,y^2) + \text{:}$$

The requirement 2° is equivalent to

$$|m(x,y)|^2 + |m(-x,y)|^2 + |m(x,-y)|^2 + |m(-x,-y)|^2 =$$

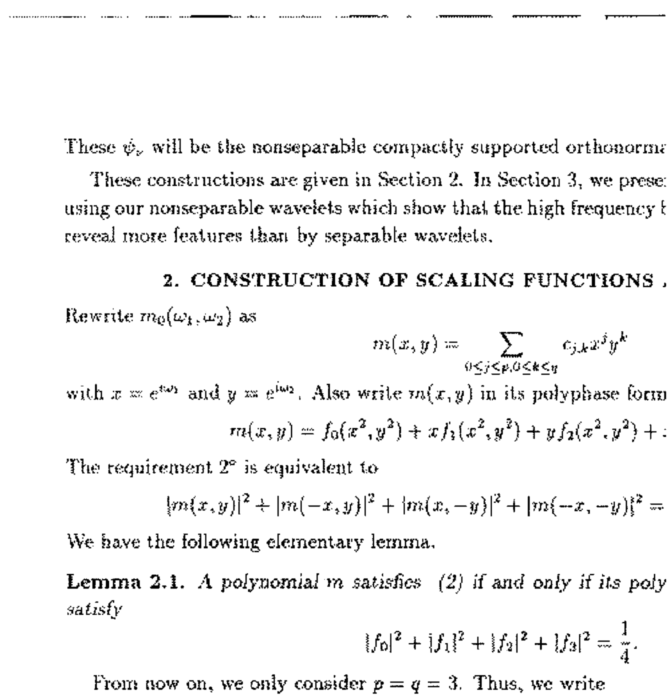
We have the following elementary lemma.

Lemma 2.1. *A polynomial m satisfies (2) if and only if its poly satisfy*

$$|f_0|^2 + |f_1|^2 + |f_2|^2 + |f_3|^2 = \frac{1}{4}.$$

From now on, we only consider $p = q = 3$. Thus, we write

(a)



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From now on, we only consider $p = q = 3$. Thus, we write

(b)

Fig. 3. (a) Text image and (b) reconstructed image after 10 : 1 compression.

Example 2.4: Let $(\theta, \xi) \in [\pi/4, 7\pi/12] \times [\pi/4, 7\pi/12]$ be fixed. Let

$$\alpha = 3\pi/4 - \arcsin\sqrt{\sin(\theta + \pi/4)\sin(\xi + \pi/4))}.$$

Set $\eta = \xi$ and $\beta = \alpha$. Then (4) is satisfied. Any θ, ξ give a filter $m(x, y)$ by (2). Using 3°, this family of filters generate orthonormal scaling functions for $(\theta, \eta) \in [\pi/4, 7\pi/12]^2$. Similar to Example 2.3,

TABLE I
PSNR IMAGE COMPRESSION COMPARISON

	Compression Ratio 10:1			Compression Ratio 15:1		
Images	Text	Lenna	Finger	Text	Lenna	Finger
Tensor Haar	24.2022	35.3078	31.5978	20.3240	32.8821	30.1455
Tensor D4	25.2776	36.5947	32.5100	21.3618	35.1661	31.0029
Tensor D6	24.8288	36.9106	32.8931	20.9433	35.5861	31.0523
Tensor 9/7	24.7949	38.1000	33.0980	20.9693	35.8116	32.0364
Non-separable	26.5489	36.4201	32.560	21.9141	34.9405	30.6138

Theorem 2.2 implies all scaling functions within the contour line $\lambda = 2$ shown in Fig. 1(b) have certain Hölder continuity.

Finally, we construct wavelets associated with the scaling ϕ. We begin with polyphase components f_0, f_1, f_2, f_3 of $m(x, y)$. Write $[f_0, f_1, f_2, f_3]^T = \mathbf{a} + x\mathbf{b} + y\mathbf{c} + xy\mathbf{d}$ with $\mathbf{a} = (a_1, a_1, a_2, a_3)^T$ and etc. Let $L = [\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ be a 4 × 4 matrix. Then letting H be the Householder transform such that HL is a lower triangular matrix, we have

$$\begin{bmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix} = H \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = HL \begin{bmatrix} 1 \\ x \\ y \\ xy \end{bmatrix} = \begin{bmatrix} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ xy \end{bmatrix}.$$

Note that by (1)

$$|\tilde{f}_0|^2 + |\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |\tilde{f}_3|^2 = |f_0|^2 + |f_1|^2 + |f_2|^2 + |f_3|^2 = \tfrac{1}{4}.$$

If $|\tilde{f}_0| = \frac{1}{2}$, then $\tilde{f}_1 = \tilde{f}_2 = \tilde{f}_2 = 0$ and we are done. Otherwise, let

$$v = [\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3]^T - \tfrac{1}{2}[1, 0, 0, 0]^T \quad \text{and}$$
$$H(v) = I_4 - \frac{2}{v^* v} v v^*$$

be a Householder matrix such that

$$H(v)[\tilde{f}_0 \quad \tilde{f}_1 \quad \tilde{f}_2 \quad \tilde{f}_3]^T = [1/2 \quad 0 \quad 0 \quad 0]^T.$$

For convenience, let $H(v)$ be either an identity matrix of size 4 × 4 if $|\tilde{f}_0| = \frac{1}{2}$ or the Householder matrix $H(v)$ above. Then we have

$$[f_0, f_1, f_2, f_3] = [\tfrac{1}{2}, 0, 0, 0]H(v)H.$$

By choosing $M(x, y) = \frac{1}{2}H(v)H$, we have $M(x, y)M^*(x, y) = \frac{1}{4}I_4$ with $[f_0, f_1, f_2, f_3]$ in the first row of $M(x, y)$. We should note that all entries of $M(x, y)$ are polynomials of x and y since $v^* v$ is a constant and H is a constant matrix. We now define polynomials $m_\nu, \nu = 0, 1, 2, 3$ with $m_0(\omega) = m(e^{i\omega_1}, e^{i\omega_2})$ as follows:

$$[m_i(\omega + \pi_j)]_{0 \le i,j \le 3} = M(e^{2i\omega_1}, e^{2i\omega_2}) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\omega_1} & 0 & 0 \\ 0 & 0 & e^{i\omega_2} & 0 \\ 0 & 0 & 0 & e^{i\omega_1 + \omega_2} \end{bmatrix}. \qquad (17)$$

Theorem 2.3: Let ϕ be the scaling function generated by m given in Theorem 2.1 and satisfying 3°. Let m_ν be trigonometric polynomials constructed in (17) above. Define the wavelets ψ_ν by $\hat{\psi}_\nu(\omega) = m_\nu(\omega/2)\hat{\phi}(\omega/2), \nu = 1, 2, 3$. Then $\{2^k \psi_\nu(2^k x - \ell_1, 2^k - \ell_2) \colon (\ell_1, \ell_2) \in \mathbf{Z}^2, k \in Z, \nu = 1, 2, 3\}$ is an orthonormal basis for $L_2(\mathbf{R}^2)$. The proof is standard using the multiresolution analysis of $L_2(\mathbf{R}^2)$ (cf. [5]). We omit the detail.

III. NUMERICAL EXPERIMENTS

We have experimented with the decomposition and reconstruction procedures using our nonseparable wavelets. In Fig. 2, we show the decompositions of Lenna and a fingerprint. We can see that the subimages in the high-frequency bands in Fig. 2 reveal more features than does by a separable filter.

We have also implemented an image compression scheme using the nonseparable wavelets in Example 2.4 and compared with the tensor product Haar, Daubechies wavelets D4 and D6, and the well-known biorthogonal wavelets with lengths nine and seven which are the wavelets for FBI fingerprint compression standard. The image compression scheme consists of multilevel wavelet decomposition, embedded zero-tree encoding and decoding [8], multilevel wavelet reconstruction, peak signal to noise ratio (PSNR) error analysis. We chose three images: a text image shown on the left of Fig. 3 besides the standard Lenna image and a finger-print image of size 512 × 512 in Fig. 2. Table I lists PSNR for these images at various compressions. We can see that our nonseparable wavelets do a significantly better job for the text image for compression ratios ten and 15.

It would be interesting to construct compactly supported scaling functions and wavelets with larger support. See [9] for a complete solution of the linear phase filters of size 6 × 6 satisfying 1°, 2°, and 6°.

REFERENCES

[1] S. D. Riemenschneider and Z. Shen, “Box splines, cardinal series, and wavelets,” in *Approximation Theory and Functional Analysis*, C. K. Chui, Ed, New York: Academic, 1991, pp. 133–149.

[2] A. Cohen and I. Daubechies, “Nonseparable dimensional wavelet bases,” *Rev. Mat. Iberoamer.*, vol. 9, pp. 51–137, 1993.

[3] J. Kovcăević and M. Vetterli, “Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $(r)^n$,” *IEEE Trans. Inform. Theory*, vol. 38, pp. 533–555, 1992.

[4] W. Lawton, “Necessary and sufficient conditions for constructing orthonormal wavelet bases,” *J. Math. Phys.*, vol. 32, pp. 57–61, 1991.

[5] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM, 1992.

[6] W. He and M. J. Lai, “Examples of bivariate nonseparable compactly supported orthonormal continuous wavelets,” *Proc. SPIE Wavelet Applications Signal Image Processing IV*, , vol. 3169, pp. 303–314, 1997.

[7] D. Colella and C. Heil, “The characterization of continuous, four coefficient scaling functions and wavelets,” *IEEE Trans. Inform. Theory*, vol. 38, pp. 876–881, 1992.

[8] A. Said and W. A. Pearlman, “A new fast and efficient image codec based on set partitioning in hierarchical trees,” *IEEE Trans. Circuits Syst. Video Technol.*, vol. 6, pp. 243–250, 1996.

[9] M. J. Lai and D. Roach, Construction of bivariate symmetric orthonormal wavelets with short support, in preparation.