Construction of Trivariate Compactly Supported Biorthogonal Box Spline Wavelets

Wenjie He† and Ming-Jun Lai *

Abstract

We give a formula for the duals of the masks associated with trivariate box spline functions. We show how to construct trivariate nonseparable compactly supported biorthogonal wavelets associated with box spline functions. The biorthogonal wavelets may have arbitrarily high regularities.

Keywords: Trivariate, Box splines, Biorthogonal Wavelets

1. Introduction

In [8], Cohen, Daubechies, and Feauveau constructed biorthogonal dual functions associated with univariate B-spline functions B_n and compactly supported biorthogonal wavelets associated with B_n. Since then, the theory of multivariate biorthogonal wavelets has been developed rapidly (cf., e.g., [6]). Since box spline functions are a natural generalization of the well-known B-spline functions, several researches have been done to construct bivariate compactly supported biorthogonal wavelets associated with box spline functions (cf. e.g., [7], [10], [16], [25], [26], [27] and [14]). Let $B_{\ell,m,n}$ be the bivariate box spline whose Fourier transform is

$$\widehat{B}_{l,m,n}(\omega) = \left(\frac{1-e^{i\omega_1}}{i\omega_1}\right)^{l} \left(\frac{1-e^{i\omega_2}}{i\omega_2}\right)^{m} \left(\frac{1-e^{i(\omega_1+\omega_2)}}{i(\omega_1+\omega_2)}\right)^{n}$$

for any positive integers ℓ, m, n and $\omega = (\omega_1, \omega_2) \in \mathbf{R}^2$. (For properties of bivariate box spline functions, see [4] and [2]. For computation of these bivariate box spline functions, see [5] and [20].) It is known that the integer translates and their dilations of a box spline function $B_{\ell,m,n}$ form a multi-resolution approximation of $L_2(\mathbf{R}^2)$ (cf. [2] or [24]). For small integers l, m, n, several different constructions of those biorthogonal wavelets were given in [7], [10], [26] and [27]. In a recent paper [14], He and Lai gave an explicit formula of the dual function $\widetilde{B}_{\ell,m,n}$ associated with box spline function $B_{\ell,m,n}$ for any integers ℓ, m, n and compactly supported biorthogonal wavelets associated with box spline function $B_{l,m,n}$ were also constructed. Those biorthogonal wavelets may have arbitrarily high regularities.

In this paper, we are interested in generalizing the explicit formula for the dual box spline functions and construction of biorthogonal box spline wavelets in [14] to the trivariate setting. That is, we shall construct the compactly supported biorthogonal wavelets

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† Department of Mathematics and Computer Science, University of Missouri, St. Louis, MO 63121-4499, His email address is he@cs.umsl.edu

* Department of Mathematics, University of Georgia, Athens, GA 30602. This author is supported by the National Science Foundation under grant DMS-9870187 and his email address is mjlai@math.uga.edu.

associated with trivariate box spline functions. Let $B_{l,m,n,p,q,r}$ be the trivariate box spline function whose Fourier transform is

$$\widehat{B}_{l,m,n,p,q,r}(\omega) = \left(\frac{1-e^{i\omega_1}}{i\omega_1}\right)^l \left(\frac{1-e^{i\omega_2}}{i\omega_2}\right)^m \left(\frac{1-e^{i\omega_3}}{i\omega_2}\right)^n \left(\frac{1-e^{i(\omega_1+\omega_2+\omega_3)}}{i(\omega_1+\omega_2+\omega_3)}\right)^p \times \\ \left(\frac{1-e^{i(\omega_2+\omega_3)}}{i(\omega_2+\omega_3)}\right)^q \left(\frac{1-e^{i(\omega_1+\omega_3)}}{i(\omega_1+\omega_3)}\right)^r$$

for any nonnegative integers ℓ, m, n, p, q, r and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbf{R}^3$. (For this choice of the direction set and other properties of trivariate box spline functions, see [2].) Without loss of generality, we may assume that all l, m, and n are positive. Since the tensor product case is not of interest here, we assume that at least one of p, q, r is not zero. It is known that $B_{l,m,n,p,q,r}$ generates a multiresolution approximation of $L_2(\mathbf{R}^3)$ (cf. [2, p. 90]). Our first step is to construct a compactly supported function $\tilde{B}_{l,m,n,p,q,r}$ generating a multiresolution approximation of $L_2(\mathbf{R}^3)$ which is a biorthogonal dual to $B_{l,m,n,p,q,r}$ in the following sense:

$$\int_{\mathbf{R}^3} B_{l,m,n,p,q,r}(\mathbf{x}-\mathbf{k})\tilde{B}_{l,m,n,p,q,r}(\mathbf{x}-\mathbf{k})d\mathbf{x} = \delta_{\mathbf{k},\mathbf{k}'} \tag{1.1}$$

for all 3 D-integers $\mathbf{k}, \mathbf{k}' \in \mathbf{Z}^3$, where $\delta_{\mathbf{j},\mathbf{k}}$ is the standard Kronecker notation defined by $\delta_{\mathbf{j},\mathbf{k}} = 0$ if $\mathbf{j} \neq \mathbf{k}$ and $\delta_{\mathbf{j},\mathbf{k}} = 1$ if $\mathbf{j} = \mathbf{k}$ and $\mathbf{Z}$ is the collection of all integers. Our second step is to construct compactly supported biorthogonal wavelets ψ_j and $\tilde{\psi}_j$ for $j = 1, \cdots, 7$ and two families of FIR filters $\{M_j, i = 1, \cdots, 7\}$ and $\{\widetilde{M}_j, j = 1, \cdots, 7\}$ with

$$\widehat{\psi}_j(\omega) = M_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}})\widehat{B}_{l,m,n,p,q}\left(\frac{\omega}{2}\right), \quad j = 1, \cdots, 7 \tag{1.2}$$

and

$$\widehat{\tilde{\psi}}_j(\omega) = \widetilde{M}_j\left(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}}\right)\widehat{\tilde{B}}_{l,m,n}\left(\frac{\omega}{2}\right), \quad j = 1, \cdots, 7 \tag{1.3}$$

such that the integer translates and their dilations of the ψ_j's and $\tilde{\psi}_j$'s form two dual Riesz bases for $L_2(\mathbf{R}^3)$ (cf. [8] for the univariate setting or [6, 19] for the multivariate setting) and the two families of masks form an exact reconstruction of synthesis/analysis filter bank which may be possibly used in data compression for 3D seismic data files.

It should be pointed out that the study of constructing compactly supported biorthogonal wavelets associated with trivariate box spline functions is not a simple generalization of the counterpart in the bivariate setting. We are only able to extend our method in [14] to the case that either $q = 0$ or $r = 0$. In this paper, we first consider trivariate box spline $B_{\ell,m,n,p,q,r}$ with $r = 0$. The case associated with $B_{\ell,m,n,p,q,r}$ with $q = 0$ and $r > 0$ follows from the case $r = 0$ and $q > 0$ immediately by the box spline symmetry

$$B_{l,m,n,p,q,0}(x_3, x_2, x_1) = B_{n,m,l,p,0,q}(x_1, x_2, x_3).$$

However, the study of the construction of biorthogonal compactly supported wavelets associated with $B_{\ell,m,n,p,q,r}$ with $q > 0, r > 0$ has to be delayed. From now on, we shall use

$$B_{\ell,m,n,p,q} := B_{\ell,m,n,p,q,0}.$$

We shall give an explicit formula for $\widehat{\widetilde{B}}_{l,m,n,p,q}$ for any given positive integers l, m, n, p and q in §2. The formula is a generalization of the counterpart in the bivariate setting in [14]. The regularities of these biorthogonal dual functions are studied in §2.2 which is based on the theory developed in [13]. General results on the regularity can be found in [9 and 17]. Although there exist some general schemes on how to find matrix extension for constructing biorthogonal wavelets (cf. [16], [26], [27] and [3]), we will give a new matrix extension scheme, which is easier to implement, for constructing M_j's and $\widetilde{M}_j$'s that lead to compactly supported biorthogonal wavelets with arbitrarily high regularities in §3. The proof of the fact that these ψ_j's and $\tilde{\psi}_j$'s generate two dual Riesz bases may be based on a straightforward generalization of the arguments for the univariate setting in [8] or based on the multivariate theory in [6] and [11]. Finally, we give several examples for small integers ℓ, m, n, p, q in §4.

§2. Construction of Compactly Supported Biorthogonal Dual Functions

§2.1 Construction of Biorthogonal Dual Masks

In the following discussion, we assume that $z = (z_1, z_2, z_3) \in \mathbf{C}^3$. We know that

$$M_0(z) = \left(\frac{1+z_1}{2}\right)^{\ell} \left(\frac{1+z_2}{2}\right)^{m} \left(\frac{1+z_3}{2}\right)^{n} \left(\frac{1+z_1z_2z_3}{2}\right)^{p} \left(\frac{1+z_2z_3}{2}\right)^{q}$$

is the refinement mask of the box spline function $B_{\ell,m,n,p,q}$. For any positive integer N, we define a bivariate polynomial

$$\mathcal{L}_N(x,y) := \sum_{k=0}^{N-1} \binom{2N-1}{k} y^k x^{N-1-k}, \tag{2.1}$$

which satisfies

$$x^N \mathcal{L}_N(x,y) + y^N \mathcal{L}_N(y,x) = (x+y)^{2N-1}. \tag{2.2}$$

Define

$$H_\tau(x,y) := \mathcal{L}_\tau \left(\frac{1+x}{2}\frac{1+y}{2}, \frac{1-x}{2}\frac{1-y}{2}\right) \tag{2.3}$$

for any positive integer τ. It follows immediately from (2.2) that

(2.4)
$$\left(\frac{1+x}{2}\right)^{\tau} \left(\frac{1+y}{2}\right)^{\tau} H_\tau(x,y) + \left(\frac{1-x}{2}\right)^{\tau} \left(\frac{1-y}{2}\right)^{\tau} H_\tau(-x,-y) = \left(\frac{1+xy}{2}\right)^{2\tau-1}.$$

Let

$$P_N(y) := \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k.$$

It is well known (see [12]) that

$$(1-y)^N P_N(y) + y^N P_N(1-y) = 1. \tag{2.5}$$

For $z=(z_1,z_2,z_3)$, we define

$$D_N(z) := (z_1z_2z_3)^{-N}\sum_{k=0}^{N-1}\binom{N-1+k}{k}(-1)^k(z_1z_2z_3)^{-k}\left(\frac{1-z_1z_2z_3}{2}\right)^{2k}.$$

Note that since each term in the summation is nonnegative, $(z_1z_2z_3)^N D_N(z)\geq 1$ for $|z_1|=|z_2|=|z_3|=1$. If we take $z_j=e^{i\omega_j}, j=1,2,3$, and let $y=\sin^2\frac{\omega_1+\omega_2+\omega_3}{2}$ in (2.5), we get

$$\left(\frac{1+z_1z_2z_3}{2}\right)^{2N} D_N(z) + \left(\frac{1-z_1z_2z_3}{2}\right)^{2N} D_N(-z) = 1, |z_j|=1, j=1,2,3 \tag{2.6}$$

for any positive integer N. Now we can define the refinement mask $\widetilde{M_0}(z)$ for $\tilde{B}_{\ell,m,n,p,q}$ as follows

$$\widetilde{M_0}(z) := \left(\frac{1+z_1^{-1}}{2}\right)^{\sigma-\ell}\left(\frac{1+z_2^{-1}}{2}\right)^{\sigma-m}\left(\frac{1+z_3^{-1}}{2}\right)^{\sigma-n}\left(\frac{1+z_1^{-1}z_2^{-1}z_3^{-1}}{2}\right)^{\rho-p}\times$$
$$\left(\frac{1+z_2^{-1}z_3^{-1}}{2}\right)^{\sigma-q} H_{\sigma,L}(z^{-1})\ \ D_{L+\eta}(z^{-1})$$

with $z^{-1}:=(z_1^{-1},z_2^{-1},z_3^{-1})$, where

$$H_{\sigma,L}(z) := \left(\frac{1+z_1}{2}\right)^{L-\sigma}\left(\frac{1+z_2z_3}{2}\right)^{L-3\sigma+1} H_\sigma(z_2,z_3)H_L(z_1,z_2z_3)$$

and the positive integers σ,ρ,L,η are so chosen that $\sigma>\max(\ell,m,n,q), \eta>(p-1)/2, \rho=2\eta+1$ and $L\geq 3\sigma-1$.

We are ready to present the main result of this subsection.

Theorem 2.1. *$\widetilde{M_0}(z)$, defined above, is a dual mask of M_0 satisfying*

$$\sum_{\ell_1,\ell_2,\ell_3\in\{0,1\}} M_0\overline{\widetilde{M_0}}\left((-1)^{\ell_1}z_1,(-1)^{\ell_2}z_2,(-1)^{\ell_3}z_3\right)=1, \quad |z_1|=|z_2|=|z_3|=1. \tag{2.7}$$

Proof: First we claim that
(2.8)

$$\sum_{\substack{\ell_1,\ell_2,\ell_3\in\{0,1\}\\(-1)^{\ell_1+\ell_2+\ell_3}=1}} M_0\overline{\widetilde{M_0}}\left((-1)^{\ell_1}z_1,(-1)^{\ell_2}z_2,(-1)^{\ell_3}z_3\right)=D_{L+\eta}(z)\left(\frac{1+z_1z_2z_3}{2}\right)^{2(L+\eta)}.$$

Indeed, the left-hand side of (2.8) can be written as

$$
\begin{aligned}
&D_{L+\eta}(z)\left(\frac{1+z_1z_2z_3}{2}\right)^{\rho}\Bigg[\left(\frac{1+z_1}{2}\right)^{L}\left(\frac{1+z_2z_3}{2}\right)^{L-2\sigma+1} H_L(z_1,z_2z_3)\times\\
&\left(\left(\frac{1+z_2}{2}\right)^{\sigma}\left(\frac{1+z_3}{2}\right)^{\sigma} H_\sigma(z_2,z_3)+\left(\frac{1-z_2}{2}\right)^{\sigma}\left(\frac{1-z_3}{2}\right)^{\sigma} H_\sigma(-z_2,-z_3)\right)\\
&+\left(\frac{1-z_1}{2}\right)^{L}\left(\frac{1-z_2z_3}{2}\right)^{L-2\sigma+1} H_L(-z_1,-z_2z_3)\times\\
&\left(\left(\frac{1-z_2}{2}\right)^{\sigma}\left(\frac{1+z_3}{2}\right)^{\sigma} H_\sigma(-z_2,z_3)+\left(\frac{1+z_2}{2}\right)^{\sigma}\left(\frac{1-z_3}{2}\right)^{\sigma} H_\sigma(z_2,-z_3)\right)\Bigg].
\end{aligned}
$$

Then (2.8) follows by using (2.4) twice for $\tau=\sigma$ and L respectively. It is easy to see (2.7) by (2.8) and (2.6). ■

We are now able to define the dual $\tilde{B}_{\ell,m,n,p,q}$ associated with box spline function $B_{\ell,m,n,p,q}$ by

$$
(2.9)\qquad \hat{\tilde{B}}_{\ell,m,n,p,q}(\omega_1,\omega_2,\omega_3)=\prod_{k=1}^{\infty}\widetilde{M}_0\left(e^{i\omega_1/2^k},e^{i\omega_2/2^k},e^{i\omega_3/2^k}\right).
$$

We first note that $\widetilde{M}_0(1,1,1)=1$ and hence $\hat{\tilde{B}}_{\ell,m,n,p,q}$ is well-defined for each $\omega\in\mathbf{R}^3$. We shall show in the next subsection that $\tilde{B}_{\ell,m,n,p,q}$ can have any high regularity by choosing integers σ, $\rho(=2\eta+1)$ and L sufficiently large. We will show that $\tilde{B}_{\ell,m,n,p,q}$ is a dual to box spline $B_{\ell,m,n,p,q}$ in the sense of (1.1) in subsection 2.3.

§2.2 Smoothness of the dual $\tilde{B}_{\ell,m,n,p,q}$

To make $\tilde{B}_{\ell,m,n,p,q}\in C^{\alpha}(\mathbf{R}^3)$ for $\alpha\geq 0$, we need to estimate the infinite product in (2.9). Note that

$$
\begin{aligned}
|H_\tau(e^{i\xi_1},e^{i\xi_2})|\leq&\sum_{k=0}^{\tau-1}\binom{2\tau-1}{k}\left|\sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2}\right|^{k}\left|\cos\frac{\xi_1}{2}\cos\frac{\xi_2}{2}\right|^{\tau-1-k}\\
\leq&\left(\sum_{k=0}^{\tau-1}\binom{2\tau-1}{k}\left|\sin^2\frac{\xi_1}{2}\right|^{k}\left|\cos^2\frac{\xi_1}{2}\right|^{\tau-1-k}\right)^{1/2}\times\\
&\left(\sum_{k=0}^{\tau-1}\binom{2\tau-1}{k}\left|\sin^2\frac{\xi_2}{2}\right|^{k}\left|\cos^2\frac{\xi_2}{2}\right|^{\tau-1-k}\right)^{1/2}\\
=&P_\tau\left(\sin^2\frac{\xi_1}{2}\right)^{1/2}P_\tau\left(\sin^2\frac{\xi_2}{2}\right)^{1/2}.
\end{aligned}
$$

The last equality can be seen in [14].

Here we need a result from [13],

$$\prod_{j=1}^{\infty} P_\tau\left(\sin^2 \frac{\xi}{2^{j+1}}\right) \le c_0(1+|\xi|)^{2\mu\tau},$$

where $\mu := \frac{\log 3}{2\log 2} < 1$. Also notice that $|D_{L+\eta}(z)| = P_{L+\eta}\left(\sin^2 \frac{\omega_1+\omega_2+\omega_3}{2}\right)$, we get

$$\begin{aligned}
&\prod_{k=1}^{\infty} \left|\widetilde{M}_0\left(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}, e^{i\frac{\omega_3}{2^k}}\right)\right| \le \left|\mathrm{sinc}\frac{\omega_1}{2}\right|^{L-\ell} \left|\mathrm{sinc}\frac{\omega_2}{2}\right|^{\sigma-m} \left|\mathrm{sinc}\frac{\omega_3}{2}\right|^{\sigma-n} \times \\
&\left|\mathrm{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-2\sigma-q+1} \left|\mathrm{sinc}\frac{\omega_1+\omega_2+\omega_3}{2}\right|^{\rho-p} C(1+|\omega_2|)^{\mu\sigma}(1+|\omega_3|)^{\mu\sigma}\times \\
&\quad (1+|\omega_1|)^{\mu L}(1+|\omega_2+\omega_3|)^{\mu L}(1+|\omega_1+\omega_2+\omega_3|)^{2\mu(L+\eta)} \\
&\le C(1+|\omega_1|)^{(\mu-1)L+\ell}(1+|\omega_2|)^{(\mu-1)\sigma+m}(1+|\omega_3|)^{(\mu-1)\sigma+n}\times \\
&\quad (1+|\omega_2+\omega_3|)^{(\mu-1)L+2\sigma+q-1}(1+|\omega_1+\omega_2+\omega_3|)^{2\mu(L+\eta)-\rho+p},
\end{aligned}$$

where $\mathrm{sinc}\,\xi := \frac{\sin\xi}{\xi}$ is the well-known sinc function.

For fixed ℓ, m, n, p, q and for any $\alpha \ge 0$, we choose σ, η, L and $\rho = 2\eta+1$, such that

$$\max((\mu-1)L+\ell, (\mu-1)\sigma+m, (\mu-1)\sigma+n) < -(\alpha+1)$$

and

$$(\mu-1)L+2\sigma+q-1 \le 0, \quad 2\mu(L+\eta)-\rho+p \le 0.$$

That is

$$\sigma > (\max(m,n)+\alpha+1)/(1-\mu), \tag{2.10}$$

$$L > \max(\ell+\alpha+1, 2\sigma+q-1)/(1-\mu), \tag{2.11}$$

$$\rho = 2\eta+1 \quad \text{with} \quad \eta \ge \frac{2\mu L+p-1}{2(1-\mu)}. \tag{2.12}$$

Therefore, we have established the following

Theorem 2.2. *Let σ, L, ρ and η be integers satisfying (2.10), (2.11) and (2.12). Then $\tilde{B}_{\ell,m,n,p,q}$ defined in (2.9) is in $C^\alpha(\mathbf{R}^3)$*

§2.3. Biorthogonality and Riesz Basis Property

We next show that $\tilde{B}_{\ell,m,n,p,q}$ defined in (2.9) is a biorthogonal dual to box spline function $B_{\ell,m,n,p,q}$ in the sense of (1.1). Indeed we have

Theorem 2.3. *For σ, L and $\rho(= 2\eta + 1)$ sufficiently large, the integer translates of $\tilde{B}_{\ell,m,n,p,q}$ form a Riesz basis for* $\operatorname{span}_{L_2(\mathbf{R}^3)} \left\{ \tilde{B}_{\ell,m,n,p,q}(\mathbf{x}-\mathbf{k}), \mathbf{k} \in \mathbf{Z}^3 \right\}$.

Proof: Mainly, we need to prove the following inequality (see e.g. [23, Chap. 2])

$$0 < A \leq \sum_{\mathbf{k} \in Z^3} \left| \hat{\tilde{B}}_{\ell,m,n,p,q}(\omega + 2\pi\mathbf{k}) \right|^2 \leq B < +\infty.$$

The second inequality follows easily from the proof of Theorem 2.2 by choosing $\alpha = 0$. The first inequality is an immediate result of Lemma 2.5, which may be proved by an extended argument in [14].

Remark 2.4. We note that the choice of $\alpha = 0$ in the proof of Theorem 2.3 is a little stronger than necessary to make $\tilde{B}_{\ell,m,n,p,q}$ to generate a Riesz basis. For specific ℓ, m, n, p and q, one may use the methods like spectual radius (cf. [11] and [17]) to get better estimate of decay of $\hat{\tilde{B}}_{\ell,m,n,p,q}(\omega)$.

Lemma 2.5. *For σ, L and $\rho(= 2\eta + 1)$ sufficiently large,*

$$\sum_{\mathbf{k} \in Z^3} \left| \hat{B}_{\ell,m,n,p,q} \hat{\tilde{B}}_{\ell,m,n,p,q} \ (\omega + 2\pi\mathbf{k}) \right|^2 \geq A > 0. \tag{2.13}$$

By noting that $\widehat{\tilde{B}}_{\ell,m,n,p,q}(0,0,0) = 1$ and $\widehat{\tilde{B}}_{\ell,m,n,p,q}$ is continuous, we can use a result in [6, Theorem 3.3] to get the following Theorem 2.6.

Theorem 2.6. *For σ, L and $\rho(= 2\eta + 1)$ sufficiently large, $\tilde{B}_{\ell,m,n,p,q}$ generates a multiresolution approximation of $L_2(\mathbf{R}^3)$, and $\tilde{B}_{\ell,m,n,p,q}$ is a biorthogonal dual to box spline $B_{\ell,m,n,p,q}$.*

Proof of Lemma 2.5.
By the periodicity and symmetry, we only need to show (2.13) for $\omega \in [0, \pi] \times [-\pi, \pi]^2$. Furthermore, it is sufficient to show that for each $\omega \in [0, \pi] \times [-\pi, \pi]^2$, there exists a multi-integer $\mathbf{k} \in \mathbf{Z}^3$, such that

$$|\hat{B}_{l,m,n,p,q} \hat{\tilde{B}}_{l,m,n,p,q}(\omega + 2\pi\mathbf{k})| \geq A > 0.$$

For simplicity, we use $\hat{B}$ and $\hat{\tilde{B}}$ to denote $\hat{B}_{l,m,n,p,q}$ and $\hat{\tilde{B}}_{l,m,n,p,q}$ respectively in the remaining of the proof. Since

$$\hat{B}\hat{\tilde{B}}(\omega) = \prod_{j=1}^{\infty} M_0 \overline{\widetilde{M}_0}(e^{i\omega_1/2^j}, e^{i\omega_2/2^j}, e^{i\omega_3/2^j}),$$

we consider $M_0 \overline{\widetilde{M}_0}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3})$. For convenience, we denote

$$\begin{aligned} |M_0 \overline{\widetilde{M}_0}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3})| =& |H^a(\omega)||H^b(\omega)||H^c(\omega)||D_{L+\eta}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3})| \\ \geq& |H^a(\omega)| \ |H^b(\omega)||H^c(\omega)|, \end{aligned} \tag{2.14}$$

since $|D_{L+\eta}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3})| \geq 1$ for $\omega_1, \omega_2, \omega_3 \in \mathbf{R}$ as pointed out earlier, where

$$(2.15) \quad H^a(\omega) := \left(\cos\frac{\omega_1}{2}\right)^L \left(\cos\frac{\omega_2}{2}\right)^\sigma \left(\cos\frac{\omega_3}{2}\right)^\sigma \left(\cos\frac{\omega_1+\omega_2}{2}\right)^{L-2\sigma+1} \left(\cos\frac{|\omega|}{2}\right)^\rho$$

with $|\omega| := \omega_1 + \omega_2 + \omega_3$, $H^b(\omega) := H_\sigma(e^{i\omega_2}, e^{i\omega_3})$ and $H^c(\omega) := H_L(e^{i\omega_1}, e^{i(\omega_2+\omega_3)})$. That is,

$$(2.16) \qquad |H^b(\omega)| = \left|\sum_{k=0}^{\sigma-1} \binom{2\sigma-1}{k} \left(-\sin\frac{\omega_2}{2}\sin\frac{\omega_3}{2}\right)^k \left(\cos\frac{\omega_2}{2}\cos\frac{\omega_3}{2}\right)^{\sigma-1-k}\right|$$

and

$$(2.17) \quad |H^c(\omega)| = \left|\sum_{k=0}^{L-1} \binom{2L-1}{k} \left(-\sin\frac{\omega_1}{2}\sin\frac{\omega_2+\omega_3}{2}\right)^k \left(\cos\frac{\omega_1}{2}\cos\frac{\omega_2+\omega_3}{2}\right)^{L-1-k}\right|.$$

It follows from (2.15) that

$$(2.18) \quad \prod_{j=1}^{\infty} |H^a(\omega/2^j)| = \left|\mathrm{sinc}\frac{\omega_1}{2}\right|^L \left|\mathrm{sinc}\frac{\omega_2}{2}\right|^\sigma \left|\mathrm{sinc}\frac{\omega_3}{2}\right|^\sigma \left|\mathrm{sinc}\frac{\omega_1+\omega_2}{2}\right|^{L-2\sigma+1} \left|\mathrm{sinc}\frac{|\omega|}{2}\right|^\rho.$$

For $\prod_{j=1}^{\infty} |H^b(\omega/2^j)|$ and $\prod_{j=1}^{\infty} |H^c(\omega/2^j)|$, we need to use the following two different kinds of estimates for each of them.

Proposition 2.7. *There exiss a real number $\delta_0 > 0$ such that*

$$(2.19) \qquad \prod_{j=1}^{\infty} |H^b\left(\frac{\omega}{2^j}\right)| \geq e^{-1} \left|\mathrm{sinc}\frac{\omega_2}{2}\mathrm{sinc}\frac{\omega_3}{2}\right|^{\sigma-1},$$

for $\omega_2 \in [-\delta_0, \delta_0], \omega_3 \in [-3\pi/2, 3\pi/2], \omega_1 \in \mathbf{R}$.

Proof: From (2.16) we have

$$(2.20) \quad \begin{aligned} |H^b\left(\frac{\omega}{2^j}\right)| &= \left|\cos\frac{\omega_2}{2^{j+1}}\cos\frac{\omega_3}{2^{j+1}}\right|^{\sigma-1} \left|\sum_{k=0}^{\sigma-1} \binom{2\sigma-1}{k} \left(-\tan\frac{\omega_2}{2^{j+1}}\tan\frac{\omega_3}{2^{j+1}}\right)^k\right| \\ &\geq \left|\cos\frac{\omega_2}{2^{j+1}}\cos\frac{\omega_3}{2^{j+1}}\right|^{\sigma-1} \left(1 - b|\tan\frac{\omega_2}{2^{j+1}}|\right) \end{aligned}$$

for some constant $b > 0$. Indeed, we write

$$\begin{aligned} &\sum_{k=0}^{\sigma-1} \binom{2\sigma-1}{k} \left(-\tan\frac{\omega_2}{2^{j+1}}\tan\frac{\omega_3}{2^{j+1}}\right)^k \\ =&1 + \tan\frac{\omega_2}{2^{j+1}}\tan\frac{\omega_3}{2^{j+1}} \sum_{k=1}^{\sigma-1} (-1)^k \binom{2\sigma-1}{k} \left(\tan\frac{\omega_2}{2^{j+1}}\tan\frac{\omega_3}{2^{j+1}}\right)^{k-1}. \end{aligned}$$

Note that $\omega_1 \in \mathbf{R}, \omega_2 \in [-\pi, \pi]^2$ and $\omega_3 \in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$ imply that $\left|\frac{\omega_2}{2^{j+1}}\right| \le \frac{3\pi}{8}$ and $\left|\frac{\omega_3}{2^{j+1}}\right| \le \frac{3\pi}{8}$. Consequently, the continuous function

$$|\tan\frac{\omega_3}{2^{j+1}} \sum_{k=1}^{\sigma-1} (-1)^k \binom{2\sigma-1}{k} \left(\tan\frac{\omega_2}{2^{j+1}} \tan\frac{\omega_3}{2^{j+1}}\right)^{k-1}|$$

has an upper bound b. Thus we have (2.20).

For $|\omega_2| \le \delta_0 := \min\{\pi, 4\tan^{-1}(1/(2b)), 1/(2b)\}$, we have

$$b|\tan\frac{\omega_2}{2^{j+1}}| \le b|\tan\frac{\omega_2}{4}| \le 1/2.$$

Thus,

$$1 - b|\tan\frac{\omega_2}{2^{j+1}}| \ge 0 \quad \text{for} \quad \omega_2 \in [-\delta_0, \delta_0], j \ge 1.$$

Since $|\tan x| \le 2|x|$ for $|x| \le \pi/4$ and $1 - |x| \ge e^{-2|x|}$ for $|x| \le 1/2$, (2.19) follows from (2.20). ■

Similarly, by (2.17), we have

Proposition 2.8. *There exists a real number $\delta_0 > 0$ such that*

$$\prod_{j=1}^{\infty} |H^c(\omega/2^j)| \ge e^{-1} \left|\text{sinc}\frac{\omega_1}{2}\text{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-1}, \tag{2.21}$$

for $\omega_1 \in [-\delta_0, \delta_0], \omega_2 + \omega_3 \in [-3\pi/2, 3\pi/2]$.

Proposition 2.9.

$$\prod_{j=1}^{\infty} |H^b(\omega/2^j)| \ge \left|\text{sinc}\frac{\omega_2}{2}\text{sinc}\frac{\omega_3}{2}\right|^{\sigma-1}, \tag{2.22}$$

for $\omega_2 \in [-2\pi, 0], \omega_3 \in [0, 2\pi]$ or for $\omega_3 \in [-2\pi, 0], \omega_2 \in [0, 2\pi]$.

Proof: For $\omega_2 \in [-2\pi, 0], \omega_3 \in [0, 2\pi]$, we have $\frac{\omega_2}{2^{j+1}} \in [-\frac{\pi}{2}, 0]$ and $\frac{\omega_3}{2^{j+1}} \in [0, \frac{\pi}{2}]$ for $j \ge 1$. Thus, each term in the summation of $H^b(\omega/2^j)$ (cf. (2.16)) is nonnegative and hence,

$$\left|H^b(\omega/2^j)\right| \ge \left|\cos\frac{\omega_2}{2^{j+1}}\cos\frac{\omega_3}{2^{j+1}}\right|^{\sigma-1}.$$

(2.22) follows immediately. ■

Similarly, we have the following

Proposition 2.10.

$$\prod_{j=1}^{\infty} |H^c(\omega/2^j)| \geq \left|\mathrm{sinc}\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-1}, \tag{2.23}$$

for $\omega_1 \in [-2\pi, 0], \omega_2+\omega_3 \in [0, 2\pi]$ *or for* $\omega_2+\omega_3 \in [-2\pi, 0], \omega_1 \in [0, 2\pi]$.

We are now ready to prove (2.13). In the following discussion, we let $\delta = \min\{\pi/3, \delta_0\}$ where δ_0 denotes the smaller number δ_0 in Propositions 2.7 and 2.8. To show (2.13) for $\omega \in [0,\pi]\times[-\pi,\pi]^2$, we divide the domain into following 4 subdomains: $[0,\pi]^3$, $[0,\pi]\times[-\pi,0]^2$, $[0,\pi]\times[-\pi,0]\times[0,\pi]$ and $[0,\pi]^2\times[-\pi,0]$. For $\omega \in [0,\pi]^3$, we consider the following three subcases.

$1°a$). For $\omega \in [0,\delta]^2 \times [0,\pi]$, using (2.18) and Propositions 2.7 and 2.8, we have

$$\begin{aligned}|\hat{B}\hat{\hat{B}}(\omega)| \geq& \prod_{j=1}^{\infty} |H^a(\frac{\omega}{2^j})| \prod_{j=1}^{\infty} |H^b(\frac{\omega}{2^j})| \prod_{j=1}^{\infty} |H^c(\frac{\omega}{2^j})| \\ \geq& e^{-2}\left|\mathrm{sinc}\frac{\omega_1}{2}\right|^{2L-1}\left|\mathrm{sinc}\frac{\omega_2}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_3}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_1+\omega_2}{2}\right|^{L-2\sigma+1}\times \\ & \left|\mathrm{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-1}\left|\mathrm{sinc}\frac{|\omega|}{2}\right|^{\rho}.\end{aligned} \tag{2.24}$$

Since $0 \leq \omega_1 \leq \delta, 0 \leq \omega_2 \leq \delta, 0 \leq \omega_3 \leq \pi$, and $0 \leq \frac{\omega_1+\omega_2}{2} \leq \delta \leq \pi/3$, all $\mathrm{sinc}\frac{\omega_1}{2}, \mathrm{sinc}\frac{\omega_2}{2}$, $\mathrm{sinc}\frac{\omega_3}{2}$ and $\mathrm{sinc}\frac{\omega_1+\omega_2}{2}$ are greater or equal to $\frac{2}{\pi}$, we have

$$\left|\mathrm{sinc}\frac{\omega_1}{2}\right|^{2L-1}\left|\mathrm{sinc}\frac{\omega_2}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_3}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_1+\omega_2}{2}\right|^{L-2\sigma+1} \geq \left(\frac{2}{\pi}\right)^{3L+2\sigma-2}.$$

Also, since $0 \leq \dfrac{\omega_2+\omega_3}{2} \leq \dfrac{\delta+\pi}{2} < \delta+\pi/2$, and $|\omega|/2 \leq \delta+\pi/2$, by the decreasing property of $\mathrm{sinc}(x)$, we get

$$\left|\mathrm{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-1}\left|\mathrm{sinc}\frac{|\omega|}{2}\right|^{\rho} \geq \left|\mathrm{sinc}(\frac{\pi}{2}+\delta)\right|^{L+\rho-1}.$$

Thus,

$$|\hat{B}\hat{\hat{B}}(\omega)| \geq e^{-2}\left(\frac{2}{\pi}\right)^{3L+2\sigma-2}\left(\mathrm{sinc}(\frac{\pi}{2}+\delta)\right)^{L+\rho-1}.$$

$1°b$). For $\omega \in [\delta,\pi]\times[0,\delta]\times[0,\pi]$, we choose $\mathbf{k} = (-1,0,0)$ and consider $|\hat{B}\hat{\hat{B}}(\omega-(2\pi,0,0))|$. As the same as above, we have, from (2.18)

$$\begin{aligned}&\prod_{j=1}^{\infty}\left|H^a(\frac{\omega-(2\pi,0,0)}{2^j})\right| = \\ &\left|\mathrm{sinc}\frac{\omega_1-2\pi}{2}\right|^{L}\left|\mathrm{sinc}\frac{\omega_2}{2}\right|^{\sigma}\left|\mathrm{sinc}\frac{\omega_3}{2}\right|^{\sigma}\left|\mathrm{sinc}\frac{\omega_1+\omega_2-2\pi}{2}\right|^{L-2\sigma+1}\left|\mathrm{sinc}\frac{|\omega|-2\pi}{2}\right|^{\rho}.\end{aligned}$$

Since $-2\pi \le \omega_1 - 2\pi \le 0$ and $0 \le \omega_2 + \omega_3 \le 2\pi$, by Proposition 2.10, we have

$$\prod_{j=1}^{\infty} |H^c(\frac{\omega - (2\pi, 0, 0)}{2^j})| \ge \left| \text{sinc}\frac{\omega_1 - 2\pi}{2} \text{sinc}\frac{\omega_2 + \omega_3}{2} \right|^{L-1}.$$

Using Proposition 2.7, we have

$$\prod_{j=1}^{\infty} |H^b(\frac{\omega - (2\pi, 0, 0)}{2^j})| \ge e^{-1} \left| \text{sinc}\frac{\omega_2}{2} \text{sinc}\frac{\omega_3}{2} \right|^{\sigma-1}.$$

As the same as subcase $1°a$), we have

$$\begin{aligned}
&|\hat{B}\hat{\hat{B}}(\omega - (2\pi, 0, 0))| \\
\ge & e^{-1} \left| \text{sinc}\frac{\omega_1 - 2\pi}{2} \right|^{2L-1} \left| \text{sinc}\frac{\omega_2}{2} \right|^{2\sigma-1} \left| \text{sinc}\frac{\omega_3}{2} \right|^{2\sigma-1} \left| \text{sinc}\frac{\omega_1 + \omega_2 - 2\pi}{2} \right|^{L-2\sigma+1} \times \\
& \left| \text{sinc}\frac{|\omega| - 2\pi}{2} \right|^{\rho} \left| \text{sinc}\frac{\omega_2 + \omega_3}{2} \right|^{L-1} \\
\ge & e^{-1} \left(\frac{2}{\pi} \right)^{4\sigma-2} \left| \text{sinc}\frac{\omega_1 - 2\pi}{2} \right|^{2L-1} \left| \text{sinc}\frac{\omega_1 + \omega_2 - 2\pi}{2} \right|^{L-2\sigma+1} \times \\
& \left| \text{sinc}\frac{|\omega| - 2\pi}{2} \right|^{\rho} \times \left| \text{sinc}\frac{\omega_2 + \omega_3}{2} \right|^{L-1} \\
\ge & e^{-1} \left| \frac{2}{\pi} \right|^{4\sigma-2} \left| \text{sinc}(\pi - \frac{\delta}{2}) \right|^{4L+\rho-2\sigma-1}
\end{aligned}$$

by using the following facts

$$\begin{aligned}
&\left| \frac{\omega_1 - 2\pi}{2} \right| \le \pi - \frac{\delta}{2}, \left| \frac{\omega_i}{2} \right| \le \frac{\pi}{2}, i = 2, 3, \left| \frac{\omega_2 + \omega_3}{2} \right| \le \frac{\pi + \delta}{2}, \\
&\left| \frac{\omega_1 + \omega_2 - 2\pi}{2} \right| \le \pi - \frac{\delta}{2}, \text{ and } \left| \frac{\omega_1 + \omega_2 + \omega_3 - 2\pi}{2} \right| \le \pi - \frac{\delta}{2}.
\end{aligned}$$

$1°c$). For $\omega \in [0, \pi] \times [\delta, \pi] \times [0, \pi]$, we choose $\mathbf{k} = (0, -1, 0)$ and consider $|\hat{B}\hat{\hat{B}}(\omega - (0, 2\pi, 0))|$. Similar to the discussion in $1°b$), we use Propositions 2.9 and 2.10 to get

$$\prod_{j=1}^{\infty} |H^c(\frac{\omega - (0, 2\pi, 0)}{2^j})| \ge \left| \text{sinc}\frac{\omega_1}{2} \text{sinc}\frac{\omega_2 + \omega_3 - 2\pi}{2} \right|^{L-1}.$$

and

$$\prod_{j=1}^{\infty} |H^b(\frac{\omega - (0, 2\pi, 0)}{2^j})| \ge \left| \text{sinc}\frac{\omega_2 - 2\pi}{2} \text{sinc}\frac{\omega_3}{2} \right|^{\sigma-1}.$$

Recall from (2.18) that

$$\prod_{j=1}^{\infty}\left|H^a\left(\frac{\omega-(0,2\pi,0)}{2^j}\right)\right| =$$
$$\left|\text{sinc}\frac{\omega_1}{2}\right|^L\left|\text{sinc}\frac{\omega_2-2\pi}{2}\right|^\sigma\left|\text{sinc}\frac{\omega_3}{2}\right|^\sigma\left|\text{sinc}\frac{\omega_1+\omega_2-2\pi}{2}\right|^{L-2\sigma+1}\left|\text{sinc}\frac{|\omega|-2\pi}{2}\right|^\rho.$$

By the same arguments as in $1^\circ b)$, we have

$$|\hat{B}\hat{\tilde{B}}(\omega-(0,2\pi,0))| \geq \left|\text{sinc}\frac{\omega_1}{2}\right|^{2L-1}\left|\text{sinc}\frac{\omega_3}{2}\right|^{2\sigma-1}\times$$
$$\left|\text{sinc}\frac{\omega_2-2\pi}{2}\right|^{2\sigma-1}\left|\text{sinc}\frac{\omega_2+\omega_3-2\pi}{2}\right|^{L-1}\left|\text{sinc}\frac{\omega_1+\omega_2-2\pi}{2}\right|^{L-2\sigma+1}\left|\text{sinc}\frac{|\omega|-2\pi}{2}\right|^\rho$$
$$\geq\left(\frac{2}{\pi}\right)^{2L+2\sigma-2}\left(\text{sinc}(\pi-\frac{\delta}{2})\right)^{2L+\rho-1}$$

since we have the following estimates

$$\left|\frac{\omega_i}{2}\right|\leq\frac{\pi}{2}, i=1,3, \left|\frac{\omega_2-2\pi}{2}\right|\leq\pi-\frac{\delta}{2}, \left|\frac{\omega_2+\omega_3-2\pi}{2}\right|\leq\pi-\frac{\delta}{2},$$
$$\left|\frac{\omega_1+\omega_2-2\pi}{2}\right|\leq\pi-\frac{\delta}{2}, \left|\frac{\omega_1+\omega_2+\omega_3-2\pi}{2}\right|\leq\pi-\frac{\delta}{2}.$$

Therefore, the discussion in subcases $1^\circ a)$–$1^\circ c)$ implies that (2.13) is true for $\omega \in [0,\pi]^3$. Similarly, we can deal with the other three subdomains. For the convenience of the interested reader, we include the details in the Appendix. ■

§3. Construction of Compactly Supported Biorthogonal Wavelets

First, we introduce a notation $A(P_0,\ldots,P_7)$ for any 8 Laurent polynomials $P_j(z), j=0,\ldots,7$ with $z=(z_1,z_2,z_3)$. $A(P_0,\ldots,P_7)$ is defined as an 8×8 matrix with columns

$$[P_j(z), P_j(-z_1,z_2,z_3), P_j(z_1,-z_2,z_3), P_j(z_1,z_2,-z_3),$$
$$P_j(-z_1,-z_2,z_3), P_j(z_1,-z_2,-z_3), P_j(-z_1,z_2,-z_3), P_j(-z)]^T,\ j=0,\ldots 7.$$

To construct biorthogonal wavelets associated with a trivariate box spline function, we need to start from the mask M_0 for the box spline function $B_{\ell,m,n,p,q}$ and the mask $\widetilde{M}_0$ for its dual function $\tilde{B}_{\ell,m,n,p,q}$ to find masks $M_1,\ldots,M_7$ and $\widetilde{M}_1,\cdots,\widetilde{M}_7$ such that

$$A(M_0,\ldots,M_7)^T A(\overline{\widetilde{M}_0},\cdots,\overline{\widetilde{M}_7}) = I_8,\quad |z_1|=|z_2|=|z_3|=1, \tag{3.1}$$

where I_8 denotes the 8×8 identity matrix. Then we can define biorthogonal wavelets ψ_j and $\tilde{\psi}_j$ for $j = 1, \cdots, 7$ by, in terms of their Fourier transforms,

$$\widehat{\psi}_j(\omega_1, \omega_2) = M_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}})\widehat{B}_{l,m,n,p,q}\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\right), \quad j = 1, \cdots, 7 \tag{3.2}$$

and

$$\widehat{\tilde{\psi}}_j(\omega_1, \omega_2) = \widetilde{M}_j\left(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}}\right)\widehat{\tilde{B}}_{l,m,n,p,q}\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\right), \quad j = 1, \cdots, 7 \tag{3.3}$$

By a result in literature (cf. [19] or [6]), these ψ_j's and $\tilde{\psi}_j$'s generate biorthogonal wavelets. That is, $\{2^\ell \psi_j(2^\ell \mathbf{x} - \mathbf{k}); \ell \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^3, j = 0, \ldots, 7\}$ and $\{2^{\ell'} \psi_{j'}(2^{\ell'} \mathbf{x} - \mathbf{k}'); \ell' \in \mathbf{Z}, \mathbf{k}' \in \mathbf{Z}^3, j' = 0, \ldots, 7\}$ constitute two dual Riesz bases for $L_2(\mathbf{R}^3)$, and

$$\int_{\mathbf{R}^3} 2^\ell \psi_j(2^\ell \mathbf{x} - \mathbf{k})\ 2^{\ell'} \tilde{\psi}_{j'}(2^{\ell'} \mathbf{x} - \mathbf{k}') d\mathbf{x} = \delta_{\ell,\ell'} \delta_{j,j'} \delta_{\mathbf{k},\mathbf{k}'}.$$

There is a matrix extension method available in the literature (cf. [26] and [27]) to find such $M_j, \widetilde{M}_j, j = 1, \cdots, 7$. However, we would like to generalize the extension method in [14] to deal with these $M_j, \widetilde{M}_j$'s. Our method does not rely on the Quillen-Suslin Theorem and does not need an orthogonal procedure as the extension method given in [26] and [27].

Our method for the construction of $M_j, \widetilde{M}_j, j = 1, \cdots, 7$ satisfying (3.1) may be divided into three steps:

Step I. Find Laurent polynomials $J_j, j = 1, \ldots, 7$, such that the determinant of the matrix $A(M_0, J_1, \cdots, J_7)$ is a non-trivial monomial. Since $M_0(z), M_0(-z_1, z_2, z_3), M_0(z_1, -z_2, z_3)$, $M_0(z_1, z_2, -z_3), M_0(-z_1, -z_2, z_3), M_0(z_1, -z_2, -z_3), M_0(-z_1, z_2, -z_3)$, and $M_0(-z)$ have no common zeros on $(\mathbf{C}\backslash\{0\})^3$, the existence of $J_1, \ldots, J_7$ is ensured by the well-known Quillen-Suslin Theorem (cf. [21] or [28]). A computation of $J_1, \cdots, J_7$ may be performed based on a general algorithm given in [22]. However, by taking advantage of the special properties of box spline functions, we shall give a concrete and elementary construction for those $J_1, \ldots, J_7$.

Step II. Compute the inverse of $A(M_0, J_1, \ldots, J_7)^T$. The inverse matrix also has the form of $A(\overline{p_0}, \overline{\widetilde{M}_1}, \cdots, \overline{\widetilde{M}_7})$ for Laurent polynomials $p_0, \widetilde{M}_1, \cdots, \widetilde{M}_7$.

Step III. Replace p_0 by $\widetilde{M}_0$ in $A(\overline{p_0}, \overline{\widetilde{M}_1}, \cdots, \overline{\widetilde{M}_7})$. The inverse of $A(\overline{\widetilde{M}_0}, \overline{\widetilde{M}_1}, \cdots, \overline{\widetilde{M}_7})$ will be the form of $A(M_0, M_1, \cdots, M_7)$. This will be clarified later.

First of all, let us give a detailed account for the first step. Let us write the mask $M_0(z)$ in the polyphase form

$$\begin{aligned} M_0(z) = & f_0(z^2) + z_1 f_1(z^2) + z_2 f_2(z^2) + z_3 f_3(z^2) \\ & + z_1 z_2 f_4(z^2) + z_2 z_3 f_5(z^2) + z_1 z_3 f_6(z^2) + z_1 z_2 z_3 f_7(z^2), \end{aligned}$$

where $z^2 := (z_1^2, z_2^2, z_3^2)$. It follows that $f_0, f_1, \ldots, f_7$ have no common zeros since

$$\begin{aligned}
&[M_0(z), M_0(-z_1, z_2, z_3), M_0(z_1, -z_2, z_3), M_0(z_1, z_2, -z_3),\\
&\ M_0(-z_1, -z_2, z_3), M_0(z_1, -z_2, -z_3), M_0(-z_1, z_2, -z_3), M_0(-z)]^T\\
&\quad = U(z)\ [f_0(z^2), \ldots, f_7(z^2)]^T
\end{aligned} \tag{3.4}$$

where

$$U(z) := \begin{bmatrix}
1 & z_1 & z_2 & z_3 & z_1z_2 & z_2z_3 & z_1z_3 & z_1z_2z_3\\
1 & -z_1 & z_2 & z_3 & -z_1z_2 & z_2z_3 & -z_1z_3 & -z_1z_2z_3\\
1 & z_1 & -z_2 & z_3 & -z_1z_2 & -z_2z_3 & z_1z_3 & -z_1z_2z_3\\
1 & z_1 & z_2 & -z_3 & z_1z_2 & -z_2z_3 & -z_1z_3 & -z_1z_2z_3\\
1 & -z_1 & -z_2 & z_3 & z_1z_2 & -z_2z_3 & -z_1z_3 & z_1z_2z_3\\
1 & z_1 & -z_2 & -z_3 & -z_1z_2 & z_2z_3 & -z_1z_3 & z_1z_2z_3\\
1 & -z_1 & z_2 & -z_3 & -z_1z_2 & -z_2z_3 & z_1z_3 & z_1z_2z_3\\
1 & -z_1 & -z_2 & -z_3 & z_1z_2 & z_2z_3 & z_1z_3 & -z_1z_2z_3
\end{bmatrix} \tag{3.5}$$

whose determinant is $4096 z_1^4 z_2^4 z_3^4$.

We have to treat the case $q = 0$ and $q > 0$ separately. We first show

Lemma 3.1. *Suppose that $q > 0$. Then the first seven polynomials $f_0, \ldots, f_6$ have no common zeros on $(\mathbf{C}\backslash\{0\})^3$.*

Proof: Suppose that $z^2 \in (\mathbf{C}\backslash\{0\})^3$ is one of the common zeros of these seven polynomials. It follows that

$$\begin{aligned}
M_0(z) = M_0(-z_1, -z_2, z_3) = M_0(-z_1, z_2, -z_3) = M_0(z_1, -z_2, -z_3) = z_1z_2z_3 f_7(z^2),\\
M_0(-z) = M_0(-z_1, z_2, z_3) = M_0(z_1, -z_2, z_3) = M_0(z_1, z_2, -z_3) = -z_1z_2z_3 f_7(z^2).
\end{aligned}$$

Thus, we have

$$(1+z_1)^\ell (1+z_2)^m (1+z_3)^n (1+z_1z_2z_3)^p (1+z_2z_3)^q \tag{3.6}$$

$$= -(1-z_1)^\ell (1+z_2)^m (1+z_3)^n (1-z_1z_2z_3)^p (1+z_2z_3)^q \tag{3.7}$$

$$= -(1+z_1)^\ell (1+z_2)^m (1-z_3)^n (1-z_1z_2z_3)^p (1-z_2z_3)^q \tag{3.8}$$

$$= (1-z_1)^\ell (1+z_2)^m (1-z_3)^n (1+z_1z_2z_3)^p (1-z_2z_3)^q \tag{3.9}$$

$$= (1-z_1)^\ell (1-z_2)^m (1+z_3)^n (1+z_1z_2z_3)^p (1-z_2z_3)^q \tag{3.10}$$

$$= -(1+z_1)^\ell (1-z_2)^m (1+z_3)^n (1-z_1z_2z_3)^p (1-z_2z_3)^q \tag{3.11}$$

$$= (1+z_1)^\ell(1-z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p(1+z_2z_3)^q \tag{3.12}$$

$$= -(1-z_1)^\ell(1-z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p(1+z_2z_3)^q. \tag{3.13}$$

It is obvious that all those terms in (3.6)—(3.13) above can not be zero simultanuously. Otherwise all polynomials $f_0, \cdots, f_7$ would have a common zero $z^2 \in (\mathbf{C}\backslash\{0\})^3$.

From (3.6) and (3.12), and (3.9) and (3.10) respectively, we have

$$(1+z_2)^m(1+z_3)^n = (1-z_2)^m(1-z_3)^n \text{ and } (1-z_2)^m(1+z_3)^n = (1+z_2)^m(1-z_3)^n.$$

Thus, $|1+z_2|^{2m} = |1-z_2|^{2m}$ and $|1-z_3|^{2n} = |1+z_3|^{2n}$. That is, z_2 and z_3 have to be purely imaginary numbers. Let us write $z_2 = bi$ and $z_3 = ci$ with $b, c \in \mathbf{R}$.

Again from (3.6) and (3.10), and (3.7) and (3.11) respectively, we have

$$(1+z_1)^\ell(1+z_2)^m(1+z_2z_3)^q = (1-z_1)^\ell(1-z_2)^m(1-z_2z_3)^q,$$
$$(1-z_1)^\ell(1+z_2)^m(1+z_2z_3)^q = (1+z_1)^\ell(1-z_2)^m(1-z_2z_3)^q.$$

It is easy to see that z_1 is a purely imaginary number. Let $z_1 = ai$ with $a \in \mathbf{R}$. By (3.8) and (3.13), we have

$$(1+ai)^\ell(1+bi)^m(1+bc)^q = (1-ai)^\ell(1-bi)^m(1-bc)^q$$

Taking the absolute value both sides, we get $|1-bc|^q = |1+bc|^q$ or $bc = 0$. That is, $b = 0$ or $c = 0$ which contradicts the assumption that $z \in (\mathbf{C}\backslash\{0\})^3$. This completes the proof. ■

Lemma 3.2. *Suppose that $q = 0$. Then the first six polynomials $f_0, \ldots, f_5$ have at most finitely many common zeros on $(\mathbf{C}\backslash\{0\})^3$.*

Proof: Suppose that $z^2 \in (\mathbf{C}\backslash\{0\})^3$ is one of the common zeros of these six polynomials. It follows that

$$\begin{aligned} M_0(z) &= -M_0(-z_1, z_2, z_3) = -M_0(z_1, z_2, -z_3) \\ &= M_0(-z_1, z_2, -z_3) = z_1z_3f_6(z^2) + z_1z_2z_3f_7(z^2) \\ M_0(-z) &= M_0(z_1, -z_2, z_3) = -M_0(-z_1, -z_2, z_3) \\ &= -M_0(z_1, -z_2, -z_3) = z_1z_3f_6(z^2) - z_1z_2z_3f_7(z^2). \end{aligned}$$

Thus, we have

$$\begin{aligned} &(1+z_1)^\ell(1+z_2)^m(1+z_3)^n(1+z_1z_2z_3)^p = -(1-z_1)^\ell(1+z_2)^m(1+z_3)^n(1-z_1z_2z_3)^p \\ &= -(1+z_1)^\ell(1+z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p = (1-z_1)^\ell(1+z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &(1+z_1)^\ell(1-z_2)^m(1+z_3)^n(1-z_1z_2z_3)^p = -(1-z_1)^\ell(1-z_2)^m(1+z_3)^n(1+z_1z_2z_3)^p = \\ &-(1+z_1)^\ell(1-z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p = (1-z_1)^\ell(1-z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p. \end{aligned}$$

The above two groups of equations can not be zero simultaneously. Without loss of generality, we assume the first group of equations is not zero. Then we can get

$$|1+z_1||1+z_3| = |1-z_1||1-z_3| \text{ and } |1-z_1||1+z_3| = |1+z_1||1-z_3|.$$

It follows that $z_1 + \overline{z_1} = 0$ and $z_3 + \overline{z_3} = 0$. That is, $z_1 = ai$ and $z_3 = ci$ with a and c real. By (3.14), we have

$$\begin{array}{rl} (1+ai)^\ell(1-acz_2)^p & = -(1-ai)^\ell(1+acz_2)^p \\ -(1-ai)^\ell(1-acz_2)^p & = (1+ai)^\ell(1+acz_2)^p \end{array},$$

which implies that

$$(1-acz_2)^{2p} = (1+acz_2)^{2p}. \tag{3.15}$$

It is easy to see that z_2 is a purely imaginary number. Let $z_2 = bi$ for some $b \in \mathbf{R}$. It follows from (3.14) that

$$\begin{array}{rl} (1+ai)^\ell(1-abci)^p & = -(1-ai)^\ell(1+abci)^p \\ -(1+ai)^\ell(1+abci)^p & = (1-ai)^\ell(1-abci)^p. \end{array} \tag{3.16}$$

Look at the complex conjugate of both sides of (3.16), one can see that $(1+ai)^\ell(1-abci)^p$ is a purely imaginary number and so is $(1+ai)^\ell(1+abci)^p$. Thus, $(1+ai)^{2\ell}(1+a^2b^2c^2)^p$ is a real number or $(1+ai)^{2\ell}$ is a real number. Consequently, $\sum_{k=0}^{\ell-1}\binom{2\ell}{2k+1}a^{2k}(-1)^k = 0$, which has only finitely many real solutions for a. Similarly there are only finitely many real solutions for c. (3.15) becomes $(1+abci)^{2p} = (1-abci)^{2p}$, which implies that $(1+abci)^{2p}$ is a real number. Obviously there are finitely many b's to make $(1+abci)^{2p}$ real. Hence, at most finitely many z's satisfy (3.14). This completes the proof of the Lemma 3.3. ■

Lemma 3.3. *There exists an 8×8 Laurent polynomial matrix $\mathcal{B}(z)$ with real coefficients such that the first column of $\mathcal{B}$ is $[f_0, f_1, \ldots, f_7]^T$ and the determinant of $\mathcal{B}$ is 1.*

Proof: We first consider the case that $q = 0$. By Lemma 3.2, we may assume that $f_0, \ldots, f_5$ have r common zeros in $(\mathbf{C}\backslash\{0\})^3$ for $r \geq 1$ (if $r = 0$, then it is trivial), which are $w_j, j = 1, \ldots, r$. Now we consider $f_6 + kf_7$ for some real number k. Since $f_0, \ldots, f_7$ have no common zero, $f_6(w_j)$ and $f_7(w_j)$ can not be equal to zero simultaneously for any $j = 1, \ldots, r$. Thus, there exists a $k_0 \neq 0$ such that $\tilde{f}_6 = f_6 + k_0 f_7$ does not vanish on all the w_j's. It follows that $f_0, \ldots, f_5, \tilde{f}_6$ have no common zero in $(\mathbf{C}\backslash\{0\})^3$. By Hilbert's Nullstellensatz Theorem, (cf. [15]), there exist polynomials $p_0, \ldots, p_6$ with real coefficients such that

$$\sum_{j=0}^{5} f_j(z)p_j(z) + \tilde{f}_6(z)p_6(z) = 1.$$

Note that

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & -k_0 \\ &&&&&&& 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \tilde{f}_6 \\ f_7 \end{bmatrix},$$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \tilde{f}_6 \\ f_7 \end{bmatrix} = \begin{bmatrix} &&&&&&& 1 \\ &&&&&& 1 & \\ &&&&& 1 && \\ &&&& 1 &&& \\ &&& 1 &&&& \\ && 1 &&&&& \\ & 1 &&&&&& \\ 1 & p_6(f_7-1) & p_5(f_7-1) & \cdots & \cdots & \cdots & p_1(f_7-1) & p_0(f_7-1) \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{f}_6 \\ f_5 \\ f_4 \\ f_3 \\ f_2 \\ f_1 \\ f_0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ \tilde{f}_6 \\ f_5 \\ f_4 \\ f_3 \\ f_2 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 &&&&&&& \\ \tilde{f}_6 & 1 &&&&&& \\ f_5 && 1 &&&&& \\ f_4 &&& 1 &&&& \\ f_3 &&&& 1 &&& \\ f_2 &&&&& 1 && \\ f_1 &&&&&& 1 & \\ f_0 &&&&&&& 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The desirable matrix $\mathcal{B}$ is the product of the three matrices above whose determinant is equal to 1.

For the case that $q > 0$, we use Lemma 3.1. In this case, we take $k = 0$, that is, $\tilde{f}_6 = f_6$. The desirable matrix $\mathcal{B}$ is the product of the last two matrices above. This completes the proof of Lemma 3.3. ■

We now give the detail of **Step II** and **III**. By Lemma 3.3 and (3.4), we can take $A(M_0, J_1, \ldots, J_7) = U(z)\mathcal{B}(z^2)$, where $U(z)$ is defined in (3.5). Since the determinant of $A(M_0, J_1, \ldots, J_7)$ is $4096 z_1^4 z_2^4 z_3^4$, it is invertible on the Laurent polynomial ring. Let $A(\overline{p_0}, \overline{\widetilde{M}_1}, \cdots, \overline{\widetilde{M}_7})$ be the inverse of $A(M_0, J_1, \ldots, J_7)^T$. Using a definition of the inverse of matrices, it is easy to see that

$$M_0(z) = \frac{1}{4096 z_1^4 z_2^4 z_3^4} \det \begin{bmatrix} \overline{\widetilde{M}}_1(-z_1, z_2, z_3) & \overline{\widetilde{M}}_1(z_1, -z_2, z_1) & \ldots & \overline{\widetilde{M}}_1(-z) \\ \overline{\widetilde{M}}_2(-z_1, z_2, z_3) & \overline{\widetilde{M}}_2(z_1, -z_2, z_3) & \ldots & \overline{\widetilde{M}}_2(-z) \\ \ldots & \ldots & \ldots & \ldots \\ \overline{\widetilde{M}}_7(-z_1, z_2, z_3) & \overline{\widetilde{M}}_7(z_1, -z_2, z_3) & \ldots & \overline{\widetilde{M}}_7(-z) \end{bmatrix}. \tag{3.17}$$

Replacing p_0 in $A(\overline{p_0}, \overline{\widetilde{M_1}}, \cdots, \overline{\widetilde{M_7}})$ by the dual mask $\widetilde{M_0}$ which is given in Theorem 2.1, we notice that $\det(A(\overline{\widetilde{M_0}}, \overline{\widetilde{M_1}}, \cdots, \overline{\widetilde{M_7}})) = 4096 z_1^4 z_2^4 z_3^4$ by the co-factor expansion of the first column, (3.17) and (2.7). Let $A(q_0, M_1, \ldots, M_7)$ be the inverse of $A(\overline{\widetilde{M_0}}, \overline{\widetilde{M_1}}, \cdots, \overline{\widetilde{M_7}})$. One can see that q_0 in $A(q_0, M_1, \ldots, M_7)$ is exactly the same as M_0 by observing that they both have the same expression of the right-hand side of (3.17). Therefore,

$$A(M_0, M_1, \ldots, M_7) A(\overline{\widetilde{M}}_0, \overline{\widetilde{M}}_1, \ldots, \overline{\widetilde{M}}_7)^T = I_8.$$

We remark here that the method used in Step II and III can be generlized to any multivariate settings.

§4. Examples

In the following, let us give some examples associated with box spline functions for small integers (ℓ, m, n, p). Based on the construction in the previous section, for case that $q \neq 0$, we only need to find polynomials $p_0, \cdots, p_6$ such that

$$p_0 f_0 + \cdots + p_6 f_6 = 1 \tag{4.1}$$

where $f_0, \cdots, f_6$ are the first 7 polyphase components of the mask for box spline function $B_{\ell,m,n,p,q,0}$. For $q = 0$, for the small integers ℓ, m, n, p, we can verify that $f_0, \cdots, f_6$ have no common zeros on $(\mathbf{C}\backslash\{0\})^3$. Thus, we can use the same method as $q \neq 0$ to construct the masks $M_1, \cdots, M_7$ and $J_2, \cdots, J_7$.

We may use the Gröbner basis method as described in [1] to compute the polynomials $p_0, \cdots, p_6$ satisfying (4.1) for polynomials $f_0, \cdots, f_6$ associated with box spline functions. (The authors wish to thank Dr. Lingyun Ma for her MATHEMATICA programs for computing $p_0, \cdots, p_6$ based on Buchberger's algorithm using the Gröbner basis.) Some outputs of those programs are given below.

Example 1. For the box spline $B_{1,1,1,1}$, we have

$$p_0 = 1/2, p_1 = -z_1^2/2, p_2 = p_3 = p_4 = 0, p_5 = 1/2, p_6 = 0.$$

Example 2. For the box spline $B_{2,2,1,1}$, we have

$$p_0 = 1/8, p_1 = -1/16, p_2 = 1/16, p_3 = -z_3^2, p_4 = 1/4, p_5 = -1/16 - z_3^2/16, p_6 = 0.$$

Example 3. For the box spline $B_{2,2,2,1}$, we have

$$p_0 = \frac{1 + 3z_3^2}{16}, p_1 = \frac{1}{2} + \frac{25z_2^2}{128} - \frac{5z_3^2}{128} p_2 = \frac{1}{2} + \frac{5z_3^2}{32},$$
$$p_3 = -\frac{17}{32} - \frac{25z_2^2}{128} - \frac{39z_3^2}{128}, p_4 = -\frac{75z_2^2}{128} - \frac{9z_3^2}{128}, p_5 = \frac{75z_2^2}{128} - \frac{63z_3^2}{128}, p_6 = \frac{3z_3^2}{32}.$$

Example 4. For box spline $B_{2,2,2,2}$, we have

$$\begin{aligned}
p_0 =& -\frac{8779}{1742528} + \frac{61137\, z_1^2}{435632} + \frac{977555\, z_2^2}{3485056} + \frac{2906109\, z_3^2}{3485056} + \frac{61137\, z_1^2\, z_3^2}{435632} + \frac{470475\, z_2^2\, z_3^2}{3485056} \\
& - \frac{54247\, z_1^2\, z_2^2\, z_3^2}{1742528} + \frac{3104437\, z_3^4}{3485056}, \\
p_1 =& -\frac{6486213}{3485056} - \frac{674691\, z_1^2}{3485056} - \frac{61137\, z_1^4}{1742528} - \frac{623455\, z_2^2}{13940224} + \frac{926105\, z_1^2\, z_2^2}{13940224} - \frac{17725709\, z_3^2}{13940224} \\
& + \frac{4845395\, z_1^2\, z_3^2}{13940224} - \frac{61137\, z_1^4\, z_3^2}{1742528} - \frac{299335\, z_2^2\, z_3^2}{13940224} - \frac{82347\, z_1^2\, z_2^2\, z_3^2}{13940224} + \frac{54247\, z_1^4\, z_2^2\, z_3^2}{3485056} \\
& - \frac{3104437\, z_3^4}{13940224} - \frac{3104437\, z_1^2\, z_3^4}{13940224}, \\
p_2 =& \frac{1915821}{871264} + \frac{172667\, z_1^2}{1742528} - \frac{2813863\, z_2^2}{13940224} - \frac{316429\, z_1^2\, z_2^2}{1742528} - \frac{193695\, z_2^4}{13940224} + \frac{13809155\, z_3^2}{13940224} \\
& - \frac{61137\, z_1^2\, z_3^2}{1742528} - \frac{2815445\, z_2^2\, z_3^2}{3485056} - \frac{61137\, z_1^2\, z_2^2\, z_3^2}{1742528} - \frac{641615\, z_2^4\, z_3^2}{13940224} - \frac{3104437\, z_3^4}{13940224} \\
& - \frac{3104437\, z_2^2\, z_3^4}{13940224}, \\
p_3 =& -\frac{16063}{108908} + \frac{63823\, z_1^2}{435632} - \frac{83985\, z_2^2}{435632} + \frac{23433\, z_3^2}{435632} - \frac{58451\, z_1^2\, z_3^2}{435632} - \frac{529635\, z_2^2\, z_3^2}{1742528} \\
& - \frac{3119483\, z_3^4}{1742528}, \\
p_4 =& \frac{-58451\, z_1^2}{217816} - \frac{14565\, z_2^2}{217816} + \frac{3191\, z_1^2\, z_2^2}{27227} - \frac{27739\, z_3^2}{871264} - \frac{42785\, z_2^2\, z_3^2}{871264}, \\
p_5 =& -\frac{456687}{1742528} - \frac{61137\, z_1^2}{871264} + \frac{2493825\, z_2^2}{6970112} - \frac{19796117\, z_3^2}{6970112} + \frac{61137\, z_1^2\, z_3^2}{871264} \\
& + \frac{656727\, z_2^2\, z_3^2}{6970112} + \frac{190301\, z_1^2\, z_2^2\, z_3^2}{1742528} + \frac{193695\, z_2^4\, z_3^2}{3485056} + \frac{3104437\, z_3^4}{6970112} + \frac{3053665\, z_2^2\, z_3^4}{3485056}, \\
p_6 =& \frac{364375}{871264} - \frac{194155\, z_1^2}{871264} + \frac{193695\, z_2^2}{6970112} + \frac{18188357\, z_3^2}{6970112} + \frac{61137\, z_1^2\, z_3^2}{871264} + \frac{641615\, z_2^2\, z_3^2}{6970112} \\
& + \frac{3104437\, z_3^4}{6970112}
\end{aligned}$$

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Appendix

We now take time and space to give a detailed account for the remaining proof of Lemma 2.5 which is a key lemma in this paper. We need to treal three subdomains $[0,\pi]^2\times[-\pi,0]$, $[0,\pi]\times[-\pi,0]\times[0,\pi]$ and $[0,\pi]\times[-\pi,0]^2$.

For $[0,\pi]^2\times[-\pi,0]$, we study the following 4 subcases:

$2^\circ a)$ For $\omega\in[0,\pi]\times[0,\pi-\delta]\times[-\pi,0]$ and $\omega_2+\omega_3\le 0$, we use Propositions 2.9 and 2.10 together with (2.18) to get (2.24) with the constant e^{-2} replaced by 1. It is easy to see the following inequalities

$$\left|\frac{\omega_i}{2}\right|\le\frac{\pi}{2}, i=1,2,3, \left|\frac{\omega_1+\omega_2}{2}\right|\le\pi-\frac{\delta}{2},$$
$$\left|\frac{\omega_1+\omega_2+\omega_3}{2}\right|\le\pi-\frac{\delta}{2}, \text{ and } \left|\frac{\omega_2+\omega_3}{2}\right|\le\frac{\pi}{2}-\frac{\delta}{2}.$$

It follows from (2.24) with the constant e^{-2} replaced by 1 that

$$|\hat{B}\hat{\tilde{B}}(\omega)|\ge\left(\frac{2}{\pi}\right)^{3L+4\sigma-4}\left(\operatorname{sinc}(\pi-\frac{\delta}{2})\right)^{L-2\sigma+\rho+1}.$$

$2^\circ b)$ For $\omega\in[0,\pi]\times[\pi-\delta,\pi]\times[-\pi,0]$ and $\omega_2+\omega_3\le 0$, we choose $\mathbf{k}=(0,-1,1)$. For $\omega_2-2\pi\in[-\pi-\delta,0]$ and $\omega_3+2\pi\in[0,2\pi]$, we can use Proposition 2.9. For $\omega_1\in[0,2\pi]$ and $\omega_2+\omega_3\le 0$, we can use Proposition 2.10. Together with (2.18), we have

$$\begin{aligned}&|\hat{B}\hat{\tilde{B}}(\omega+(0,-2\pi,2\pi))|\\ &\ge\left|\operatorname{sinc}\frac{\omega_1}{2}\right|^{2L-1}\left|\operatorname{sinc}\frac{\omega_2-2\pi}{2}\right|^{2\sigma-1}\left|\operatorname{sinc}\frac{\omega_3+2\pi}{2}\right|^{2\sigma-1}\times\\ &\left|\operatorname{sinc}\frac{\omega_1+\omega_2-2\pi}{2}\right|^{L-2\sigma+1}\left|\operatorname{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-1}\left|\operatorname{sinc}\frac{|\omega|}{2}\right|^{\rho}.\end{aligned}\tag{2.25}$$

The following inequalities can be verified easily:

$$\left|\frac{\omega_1}{2}\right|\le\frac{\pi}{2}, \left|\frac{\omega_2+\omega_3}{2}\right|\le\frac{\pi}{2}, \left|\frac{\omega_2-2\pi}{2}\right|\le\frac{\pi+\delta}{2},$$
$$\left|\frac{\omega_1+\omega_2-2\pi}{2}\right|\le\frac{\pi+\delta}{2}, \left|\frac{\omega_3+2\pi}{2}\right|\le\frac{\pi}{2}+\frac{\delta}{2}, \left|\frac{\omega_1+\omega_2+\omega_3}{2}\right|\le\frac{\pi}{2},$$

where we have used the assumption that $\omega_2+\omega_3\le 0$. Indeed, $\omega_3\le-\omega_2\le-\pi+\delta$. Thus, $\omega_3+2\pi\in[\pi,\pi+\delta]$. It thus follows from (2.25) that

$$|\hat{B}\hat{\tilde{B}}(\omega+(0,-2\pi,2\pi))|\ge\left(\frac{2}{\pi}\right)^{3L+\rho-2}\left(\operatorname{sinc}(\frac{\pi+\delta}{2})\right)^{L+2\sigma+\rho-1}.$$

$2^\circ c)$ For $\omega \in [0,\delta] \times [0,\pi] \times [-\pi, 0]$ and $\omega_2 + \omega_3 \geq 0$, we use Propositions 2.9 and 2.8 together with (2.18) to get (2.24) with the constant e^{-2} replaced by e^{-1}. It is easy to see the following inequalities:

$$\left|\frac{\omega_i}{2}\right| \leq \frac{\pi}{2}, i = 1,2,3, \left|\frac{\omega_1+\omega_2}{2}\right| \leq \frac{\pi+\delta}{2},$$
$$\left|\frac{\omega_1+\omega_2+\omega_3}{2}\right| \leq \frac{\pi+\delta}{2}, \text{ and } \left|\frac{\omega_2+\omega_3}{2}\right| \leq \frac{\pi}{2}.$$

It thus follows from (2.24) (with the constant e^{-1} instead of e^{-2}) that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \geq e^{-1}\left(\frac{2}{\pi}\right)^{3L+4\sigma-4}\left(\operatorname{sinc}(\frac{\pi+\delta}{2})\right)^{L-2\sigma+\rho+1}.$$

$2^\circ d)$ For $\omega \in [\delta,\pi] \times [0,\pi] \times [-\pi,0]$ and $\omega_2+\omega_3 \geq 0$, we choose $\mathbf{k} = (-1,0,0)$. Since $\omega_1 - 2\pi \in [-2\pi, 0]$, we use Proposition 2.10 and 2.9. Together with (2.18), we have

$$\begin{aligned} &|\hat{B}\hat{\tilde{B}}(\omega + (-2\pi,0,0))| \\ (2.26)\qquad &\geq \left|\operatorname{sinc}\frac{\omega_1-2\pi}{2}\right|^{2L-1}\left|\operatorname{sinc}\frac{\omega_2}{2}\right|^{2\sigma-1}\left|\operatorname{sinc}\frac{\omega_3}{2}\right|^{2\sigma-1}\left|\operatorname{sinc}\frac{\omega_1+\omega_2-2\pi}{2}\right|^{L-2\sigma+1}\times \\ &\left|\operatorname{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-1}\left|\operatorname{sinc}\frac{|\omega|-2\pi}{2}\right|^{\rho}. \end{aligned}$$

The following inequalities

$$\left|\frac{\omega_1-2\pi}{2}\right| \leq \pi - \frac{\delta}{2}, \left|\frac{\omega_i}{2}\right| \leq \frac{\pi}{2}, i = 2,3$$
$$\left|\frac{\omega_2+\omega_3}{2}\right| \leq \frac{\pi}{2}, \left|\frac{\omega_1+\omega_2-2\pi}{2}\right| \leq \pi - \frac{\delta}{2}, \left|\frac{\omega_1+\omega_2+\omega_3-2\pi}{2}\right| \leq \pi - \frac{\delta}{2}$$

can be verified easily. It thus follows from (2.26) that

$$|\hat{B}\hat{\tilde{B}}(\omega + (-2\pi,0,0))| \geq \left(\frac{2}{\pi}\right)^{L+4\rho-3}\left(\operatorname{sinc}(\pi - \frac{\delta}{2})\right)^{3L-2\sigma+\rho}.$$

The above 4 subcases imply that (2.13) holds for $\omega \in [0,\pi]^2 \times [-\pi,0]$. Next we consider $[0,\pi] \times [-\pi,0]^2$. We have 4 subcases to study again.

$3^\circ a)$ For $\omega \in [0,\delta] \times [-\delta,0] \times [-\pi,0]$, we use Propositions 2.7 and 2.8 together with (2.18) to get (2.24). Clearly, we have

$$\left|\frac{\omega_i}{2}\right| \leq \frac{\pi}{2}, i = 1,2,3, \left|\frac{\omega_1+\omega_2}{2}\right| \leq \frac{\pi}{2},$$
$$\left|\frac{\omega_2+\omega_3}{2}\right| \leq \frac{\pi+\delta}{2}, \left|\frac{\omega_1+\omega_2+\omega_3}{2}\right| \leq \frac{\pi+\delta}{2}.$$

It follows that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \geq e^{-2}\left(\frac{2}{\pi}\right)^{3L+2\sigma-2}\left(\mathrm{sinc}(\frac{\pi}{2}+\delta)\right)^{L+\rho-1}.$$

$3°b)$ For $\omega \in [\delta,\pi]\times[-\delta,0]\times[-\pi,0]$, We use Propositions 2.7 and 2.10. Together with (2.18), we get (2.24). As the same as above, we have

$$\left|\frac{\omega_i}{2}\right| \leq \frac{\pi}{2}, i=1,2,3, \left|\frac{\omega_1+\omega_2}{2}\right| \leq \frac{\pi+\delta}{2},$$
$$\left|\frac{\omega_2+\omega_3}{2}\right| \leq \frac{\pi+\delta}{2}, \left|\frac{\omega_1+\omega_2+\omega_3}{2}\right| \leq \frac{\pi}{2}.$$

It follows that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \geq e^{-2}\left(\frac{2}{\pi}\right)^{3L+2\sigma+\rho-2}\left(\mathrm{sinc}(\frac{\pi}{2}+\delta)\right)^{L-1}.$$

$3°c)$ For $\omega \in [\tilde{\delta},\pi]\times[-\pi,-\delta]\times[-\pi,0]$ with $0<\tilde{\delta}<\delta$ to be determined in the next subcase, we choose $\mathbf{k}=(-1,1,0)$. Then $\omega_1-2\pi \in [-2\pi+\tilde{\delta},-\pi]$ and $\omega_2+2\pi \in [\pi,2\pi-\delta]$. We use Propositions 2.9 and 2.10. Together with (2.18), we have

$$\begin{aligned}
&|\hat{B}\hat{\tilde{B}}(\omega+(-2\pi,2\pi,0))| \\
&\geq \left|\mathrm{sinc}\frac{\omega_1-2\pi}{2}\right|^{2L-1}\left|\mathrm{sinc}\frac{\omega_2+2\pi}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_3}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_1+\omega_2}{2}\right|^{L-2\sigma+1}\times \\
&\left|\mathrm{sinc}\frac{\omega_2+\omega_3+2\pi}{2}\right|^{L-1}\left|\mathrm{sinc}\frac{|\omega|}{2}\right|^{\rho}.
\end{aligned} \tag{2.27}$$

Clearly, we have

$$\left|\frac{\omega_1-2\pi}{2}\right| \leq \pi-\frac{\tilde{\delta}}{2}, \left|\frac{\omega_2+2\pi}{2}\right| \leq \pi-\frac{\delta}{2}, \left|\frac{\omega_3}{2}\right| \leq \frac{\pi}{2}$$
$$|\omega_1+\omega_2| \leq \frac{\pi}{2}, |\omega_2+\omega_3+2\pi| \leq \pi-\frac{\delta}{2}, |\omega_1+\omega_2+\omega_3| \leq \pi-\frac{\tilde{\delta}}{2}.$$

It follows from (2.27) that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \geq e^{-2}\left(\frac{2}{\pi}\right)^{2\sigma-1}\left(\mathrm{sinc}(\pi-\frac{\tilde{\delta}}{2})\right)^{4L+\rho-2}.$$

$3°d)$ The remaining subcase is $\in [0,\tilde{\delta}]\times[-\pi,-\delta]\times[-\pi,0]$ with $\tilde{\delta}$ as in $3°c)$. We shall choose $\mathbf{k}=(0,1,0)$. Then $(\omega_1,\omega_2+2\pi,\omega_3) \in [0,\tilde{\delta}]\times[\pi,2\pi-\delta]\times[-\pi,0]$. We now take time to determine $\tilde{\delta}$. Consider $H^c(0,(\omega_2+2\pi)/2,\omega_3/2)$. We have

$$|H^c(0,(\omega_2+2\pi)/2,\omega_3/2)| \geq (\cos(\pi/2-\delta/4))^{L-1} = \sin(\delta/4)^{L-1}.$$

Since $H^c(\omega_1/2, (\omega_2+2\pi)/2, \omega_3/2)$ is a continuous function in a closed domain, there exists a $\tilde{\delta} > 0$ such that

$$|H^c(\omega_1/2, (\omega_2+2\pi)/2, \omega_3/2)| \geq \frac{1}{2}\sin(\delta/4)^{L-1}$$

for $\omega_1 \in [0, \tilde{\delta}]$ and $\omega_2 + 2\pi \in [\pi, 2\pi - \delta]$ and $\omega_3 \in [-\pi, 0]$. For convenience, we let $\tilde{\delta} < \delta/2$. Thus, we use the same proof as that for Proposition 2.8 to get

$$\prod_{j=1}^{\infty} |H^c(\frac{\omega}{2^j})| \geq \frac{1}{2}\sin(\delta/4)^{L-1}e^{-1}\left|\mathrm{sinc}\frac{\omega_1}{4}\mathrm{sinc}\frac{\omega_2+\omega_3}{4}\right|^{L-1}.$$

Using Proposition 2.9 and (2.18), we get

(2.28)

$$\begin{aligned}
&|\hat{B}\hat{\tilde{B}}(\omega + (0, 2\pi, 0))| \\
&\geq \left|\mathrm{sinc}\frac{\omega_1}{2}\right|^L \left|\mathrm{sinc}\frac{\omega_1}{4}\right|^{L-1} \left|\mathrm{sinc}\frac{\omega_2+2\pi}{2}\right|^{2\sigma-1} \left|\mathrm{sinc}\frac{\omega_3}{2}\right|^{2\sigma-1} \left|\mathrm{sinc}\frac{\omega_1+\omega_2+2\pi}{2}\right|^{L-2\sigma+1} \times \\
&\left|\mathrm{sinc}\frac{\omega_2+\omega_3+2\pi}{4}\right|^{L-1} \left|\mathrm{sinc}\frac{|\omega|+2\pi}{2}\right|^{\rho} \frac{e^{-1}}{2}\left(\sin\frac{\delta}{4}\right)^{L-1}.
\end{aligned}$$

Clearly, we have

$$\left|\frac{\omega_i}{2}\right| \leq \frac{\pi}{2}, i = 1, 3, \text{ and } \left|\frac{\omega_1+\omega_2+2\pi}{2}\right| \leq \pi - \frac{\delta - \tilde{\delta}}{2}.$$

Similarly, we have

$$\left|\frac{\omega_2+\omega_3+2\pi}{2}\right| \leq \pi - \frac{\delta}{2} \text{ and } \left|\frac{\omega_1+\omega_2+\omega_3+2\pi}{2}\right| \leq \pi - \frac{\delta-\tilde{\delta}}{2}.$$

It thus follows from (2.28) that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \geq e^{-1}\left(\frac{2}{\pi}\right)^{2\sigma-1}\left(\mathrm{sinc}(\pi - \frac{\delta-\tilde{\delta}}{2})\right)^{L-2\sigma+\rho+1}\left(\sin(\frac{\delta}{4})\right)^{L-1}.$$

These complete the proof of (2.13) for the subdomain $[0, \pi] \times [-\pi, 0]^2$.

Finally, we consider the subdomain $[0, \pi] \times [-\pi, 0] \times [0, \pi]$. As above, we have 4 subcases to deal with.

4°*a*) For $\omega \in [0, \pi] \times [-\pi, 0] \times [0, \pi]$ with $\omega_2 + \omega_3 \leq 0$, we use Propositions 2.9 and 2.10. With (2.18), we get (2.24) without the constant e^{-2}. It is easy to see that

$$\left|\frac{\omega_i}{2}\right| \leq \frac{\pi}{2}, i = 1, 2, 3, \left|\frac{\omega_1+\omega_2}{2}\right| \leq \frac{\pi}{2},$$

$$\left|\frac{\omega_2+\omega_3}{2}\right| \leq \frac{\pi}{2}, \left|\frac{\omega_1+\omega_2+\omega_3}{2}\right| \leq \frac{\pi}{2}$$

because of the assumption $\omega_2+\omega_3 \le 0$. Therefore,

$$|\hat{B}\hat{\tilde{B}}(\omega)| \ge \left(\frac{2}{\pi}\right)^{4L+2\sigma+\rho-2}.$$

$4°b)$ For $\omega \in [0,\delta]\times[-\pi,0]\times[0,\pi]$ with $\omega_2+\omega_3>0$, we use Propositions 2.8 and 2.9. With (2.18), we have (2.24). Clearly, we have

$$\left|\frac{\omega_i}{2}\right| \le \frac{\pi}{2}, i=1,2,3, \left|\frac{\omega_1+\omega_2}{2}\right| \le \frac{\pi}{2},$$
$$\left|\frac{\omega_2+\omega_3}{2}\right| \le \frac{\pi}{2}, \left|\frac{\omega_1+\omega_2+\omega_3}{2}\right| \le \frac{\pi+\delta}{2}.$$

It follows that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \ge e^{-2}\left(\frac{2}{\pi}\right)^{4L+2\sigma-2}\left(\mathrm{sinc}\frac{\pi+\delta}{2}\right)^{\rho}.$$

$4°c)$ For $\omega \in [\delta,\pi]\times[-\delta/2,0]\times[0,\pi]$ with $\omega_2+\omega_3>0$, we choose $\mathbf{k}=(-1,0,0)$. Since $\omega_1-2\pi \in [-2\pi+\delta,0]$ and $\omega_2+\omega_3>0$, we can use Proposition 2.10. Clearly, we can use either Proposition 2.9 or 2.7. Together with (2.18), we have (2.26). The following inequalities can be verified easily.

$$\left|\frac{\omega_1-2\pi}{2}\right| \le \pi-\frac{\delta}{2}, \left|\frac{\omega_i}{2}\right| \le \frac{\pi}{2}, i=2,3, \left|\frac{\omega_1+\omega_2-2\pi}{2}\right| \le \pi-\frac{\delta}{4},$$
$$\left|\frac{\omega_2+\omega_3}{2}\right| \le \frac{\pi}{2}, \left|\frac{\omega_1+\omega_2+\omega_3-2\pi}{2}\right| \le \pi-\frac{\delta}{2}.$$

Here, we have used the fact $\omega_2+\omega_3>0$ in the last inequality. It follows from (2.26) that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \ge e^{-2}\left(\frac{2}{\pi}\right)^{L+4\sigma-3}\left(\mathrm{sinc}(\pi-\frac{\delta}{4})\right)^{3L+\rho-2\sigma}.$$

$4°d)$ For $\omega \in [\delta,\pi]\times[-\pi,-\delta/2]\times[0,\pi]$ and $\omega_2+\omega_3>0$, we choose $\mathbf{k}=(-1,1,-1)$. Since $\omega_1-2\pi \in [-2\pi+\delta,-\pi], \omega_2+2\pi \in [\pi, 2\pi-\delta/2]$ and $\omega_3-2\pi \in [-2\pi,-\pi]$, we can use Propositions 2.9 and 2.10 since $\omega_2+\omega_3 \ge 0$. Together with (2.18), we have

(2.28)
$$\begin{aligned}&|\hat{B}\hat{\tilde{B}}(\omega+(-2\pi,2\pi,-2\pi))|\\ \ge &\left|\mathrm{sinc}\frac{\omega_1-2\pi}{2}\right|^{2L-1}\left|\mathrm{sinc}\frac{\omega_2+2\pi}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_3-2\pi}{2}\right|^{2\sigma-1}\left|\mathrm{sinc}\frac{\omega_1+\omega_2}{2}\right|^{L-2\sigma+1}\times\\ &\left|\mathrm{sinc}\frac{\omega_2+\omega_3+2\pi}{2}\right|^{L-1}\left|\mathrm{sinc}\frac{|\omega|-2\pi}{2}\right|^{\rho}.\end{aligned}$$

Clearly, we have

$$\left|\frac{\omega_1-2\pi}{2}\right| \le \pi-\frac{\delta}{2}, \left|\frac{\omega_2+2\pi}{2}\right| \le \pi-\frac{\delta}{4}, \left|\frac{\omega_2+\omega_3}{2}\right| \le \frac{\pi}{2},$$
$$\left|\frac{\omega_1+\omega_2}{2}\right| \le \frac{\pi}{2}, \left|\frac{\omega_1+\omega_2+\omega_3-2\pi}{2}\right| \le \pi-\frac{\delta}{2}, \left|\frac{\omega_3-2\pi}{2}\right| \le \pi-\frac{\delta}{2}$$

where we have used the assumption $\omega_2 + \omega_3 > 0$. Indeed, $\omega_3 \geq -\omega_2 \geq \delta/2$. Thus, $\omega_3 - 2\pi \in [\delta/2 - 2\pi, -\pi]$. It therefore follows that

$$|\hat{B}\hat{\tilde{B}}(\omega)| \geq e^{-2} \left(\frac{2}{\pi}\right)^{2L-2\sigma} \left(\mathrm{sinc}(\pi - \frac{\delta}{4})\right)^{2L+\rho+4\sigma-2} .$$

All the above detailed discussions furnish the proof of Lemma 2.5 . ■