Recent Advances in the Classification of Two Dimensional Polynomial Hypergoups

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Abstract

In one dimension the classification of continuous polynomial hypergroups is well understood. Although much is known in higher dimensions, there is as yet no semblance of a complete theory. Even in two dimensions there are many open questions. An important tool the development of the two dimensional theory has been the discovery of families of orthogonal polynomials which are not products of two one dimensional examples. In this respect the families described by Tom Koornwinder in a series of four papers in Indagationes Mathematicae in 1974 have been invaluable. Enough is now understood from these examples to understand why a two dimensional characterization has been difficult, and why a three dimensional theory may not be practical.

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10. Non-Hermitian polynomial hypergroups in two variables

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1. Notation

- Let $H \subset \mathbf{R}^2$ be a compact set of full measure, $M(H)$ be the real-valued Borel measures supported on H.

- Let $\mathbf{N}_0 = \{0, 1, 2, \cdots\}$.

- $D = \{z \mid |z| \leq 1\}$.

- $I = [-1, 1]$

- $x = (x_1, x_2) \in \mathbf{R}^2$

- $k = (k_1, k_2) \in \mathbf{N}_0^2$ and $|k| = k_1 + k_2$
- $x^k = x_1^{k_1} x_2^{k_2}$
- A polynomial of degree $|n|$ in 2 variables is written

$$p_n(x) = \sum_{|k| \leq |n|} a_k x^k.$$

Where it is implied that $a_n \neq 0$.

The Sociology of polynomials

We will be interested in collections of polynomials of the form

$$\mathcal{P} = \{p_n(x)\}_{n \in \Lambda},$$

where Λ could be a finite set, an infinite set, or even all of $\mathbf{N}_0^2$. The collections may or may not be orthogonal.

2. $\mathcal{P}$ is **algebraically complete** if

1. $\mathcal{P}$ is linearly independent as a vector space.

2. For each k in $\mathbf{N}_0$, the elements of $\mathcal{P}$ span all polynomials of degree k.

For polynomials in a single variable, this is trivial, for polynomials in 2 variables, there must be $k+1$ polynomials of degree k in $\mathcal{P}$.

3. A collection of product formulas

The elements of $\mathcal{P}$ satisfy a (positive) **product formula** on H if for each $s, t \in H$, there exists a (positive) measure $\mu_{s,t} \in M(H)$ such that

$$p_n(s)p_n(t) = \int_H p_n(x) d\mu_{s,t}(x).$$

Characterize the families $\mathcal{P}$ which

1. are algebraically complete,
2. and have a positive product formula.

A better product formula.

$\mathcal{P}$ satisfies a **strong product formula** on H if

1. $\mathcal{P}$ satisfies a positive product formula on H.

2. There exists an element $e \in H$, called the **identity** such that:

   (a) $\mu_{z,e} = \delta_z$

   (b) $lim_{w \to e}\, diam(supp(\mu_{z,w})) = 0$ with $z, w \in H$.

Product formulas lead to measure algebras.

1. Product formulas can be used to define a convolution, $*$, on $M(H)$.

2. Positive product formulas make $M(H)$ into a Banach algebra $(M(H), *)$.

3. Strong product formulas give this algebra an identity, δ_e, and some control over the structure of H.

4. If $(M(H), *)$ satisfies some additional conditions, then the measure algebra becomes a **hypergroup**.

$(M(H), *)$ is a **compact hypergroup** if the following are satisfied

1. If μ and ν are probability measures, so is $\mu * \nu$.

2. There is an element $e \in H$ such that for all $\mu \in M(H)$
$$\mu * \delta_e = \delta_e * \mu = \mu.$$

3. There exists an involution $x \to x^{\curlyvee}$ such that $e \in supp(\delta_x * \delta_y)$ if and only if $y = x^{\curlyvee}$.

4. $(\mu * \nu)^{\curlyvee} = \nu^{\curlyvee} * \mu^{\curlyvee}$.

5. The mapping $(x, y) \to supp(\delta_x * \delta_y)$ is continuous from $H \times H$ into the compact subsets of H topologized with the Hausdorff metric.

6. The mapping $(\mu, \nu) \to \mu * \nu$ is weak-$*$ continuous.

Convolution and characters

The product formula measures can be used to define the convolution of point masses:

$$\delta_s * \delta_t \equiv \mu_{s,t}$$

and typically the support of $\mu_{s,t}$ is a set, not a point. And this can be extended to all of the measures in $M(H)$.

ϕ is a **character** for $(M(H), *)$ if ϕ is continuous on H and

$$\int_H \phi \, d(\delta_s * \delta_t) = \phi(s)\phi(t)$$

for all $s, t \in H$.

5. Compact polynomial hypergroups

If the family $\mathcal{P}$ satisfies a product formula which generates a hypergroup, $(M(H), *)$, then the product formula is called a **hypergroup product formula** and the elements of $\mathcal{P}$ are characters of this measure algebra.

If $\mathcal{P}$ is algebraically complete, then $M(H), *)$ is called a **compact polynomial hypergroup**.

6. Hermitian vs. non-Hermitian

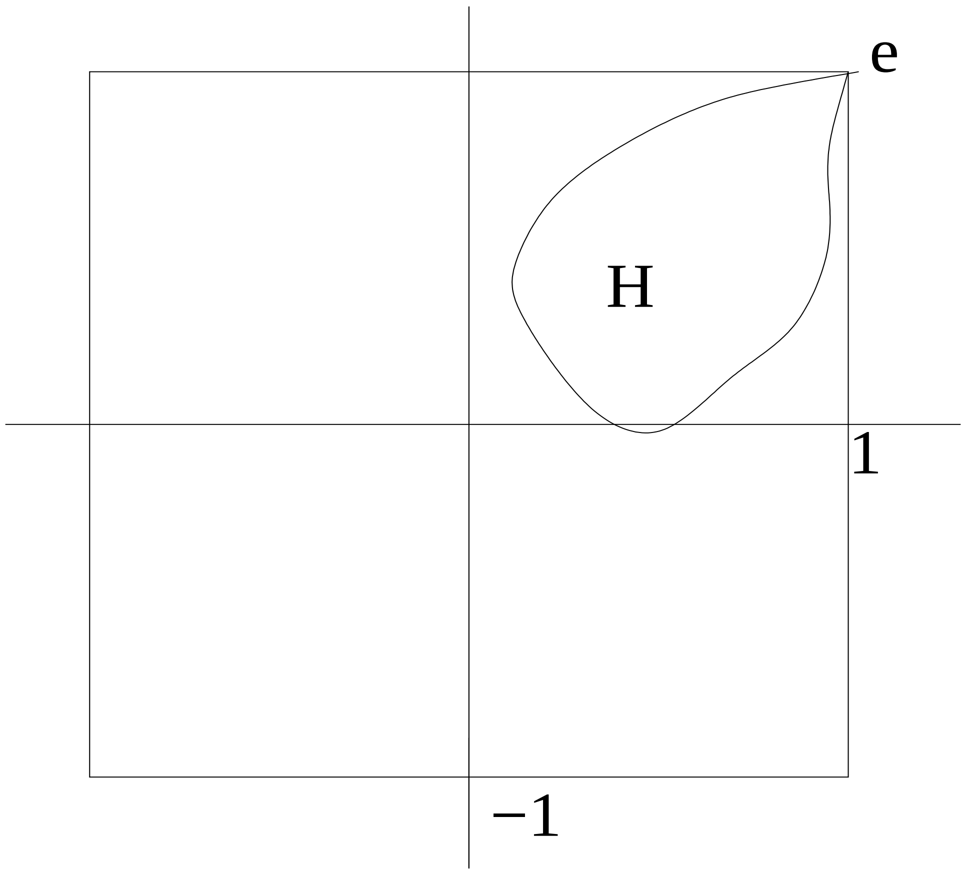
If $\mathcal{P}$ satisfies a hypergroup product formula on $(M(H), *)$, then either: $x^{\curlyvee} = x$, and the hypergroup is called **Hermitian**, otherwise the hypergroup is called **non-Hermitian**.

The canonical Hermitian hypergroup

Theorem (CS95). Any Hermitian hypergroup which has exactly two distinct non-constant characters which are first degree polynomials is linearly equivalent to $(M(H), *)$ where

1. $e = (1,1) \in H$
2. $(x,y)^{\curlyvee} = (x,y)$
3. $H \subseteq I^2$
4. $\phi_{1,0}(x,y) = x, \quad \phi_{0,1}(x,y) = y.$

Hermitian Case



Theorem for Hermitian hypergroups

The Jacobi polynomials, $\mathcal{P}^{\alpha,\beta} = \{P_n^{\alpha,\beta}(x)\}$, have a product formula for one range of values of the indices, a positive product formula in a slightly smaller range (Gas71),

$$E_J = \{(\alpha, \beta) : \alpha \geq \beta > -1, \text{ and either } \beta \geq -\frac{1}{2} \text{ or } \alpha + \beta \geq 0\}.$$

If $(M(H), *)$ is a canonical Hermitian hypergroup (as above), then the elements of $\mathcal{P}$ are products of Jacobi polynomials with indices in E_J if and only if $H \cap (I * \{1\})$ and $H \cap (\{1\} * I)$ are both infinite sets.

The canonical non-Hermitian hypergroup

Theorem (CS95). Any non-Hermitian hypergroup which has exactly two distinct non-constant characters which are first degree polynomials is linearly equivalent to $(M(H), *)$, where

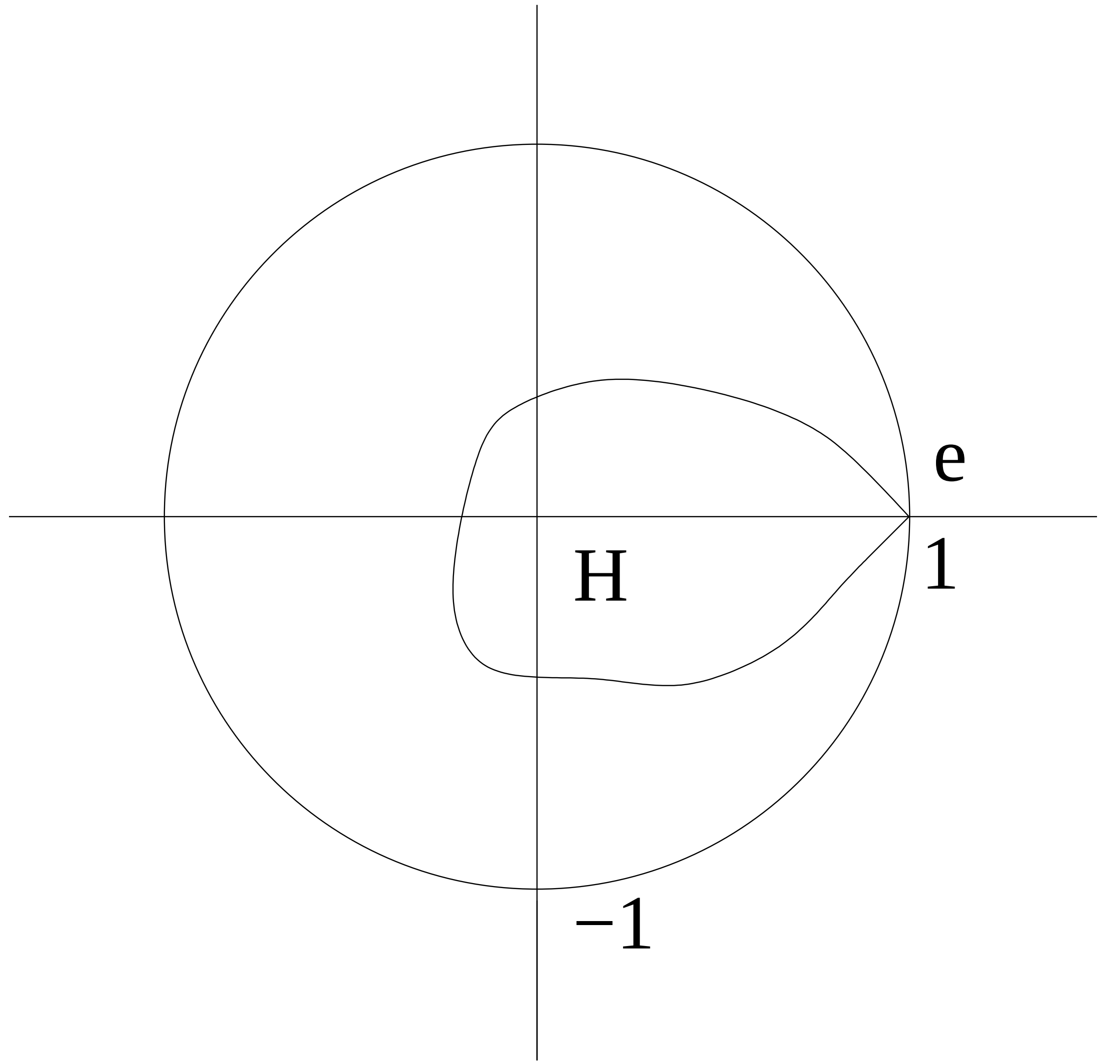
1. $e = (1, 0) \in H$

2. $(x, y)^{\curlyvee} = (x, -y)$

3. $H \subseteq D$

4. $\phi_{1,0}(x, y) = (x, y), \quad \phi_{0,1}(x, y) = (x, -y)$.

Non−Hermitian Case



Theorem for non-Hermitian hypergroups

If $(M(H), *)$ is a canonical non-Hermitian hypergroup (as above), then the elements of $\mathcal{P}$ are disk polynomials, D_γ, for $\gamma > 0$ if and only if the identity element $e = (1, 0)$ is a 2-dimensional accumulation point of H, and $\{x \in H : |x| = 1\}$ contains at least seven points.

7. Issues of orthogonality

In a single variable, an algebraically complete family of orthogonal polynomials is uniquely determined up to multiplicative constants. In several variables this is no longer true. This ambiguity is the source of some confusion, and stems from the fact that the orthogonal polynomials that emerge depend on the order chosen for the original sequence of polynomials that is used to create the orthogonal sequence.

Two polynomials are orthogonal

There is no disagreement about what is meant for two polynomial characters to be orthogonal on a given set with respect to a given weight function (or measure).

Assume that $H \subset R^2$, and the positive function w is supported on H. Then the polynomial characters p, q are **orthogonal on** H **with respect to the weight** w if

$$\langle p, q \rangle \equiv \int_H p(x) q(x)^\curlyvee w(x) dx = 0 \qquad p \neq q.$$

Families of orthogonal polynomials

In several variables, there are at least three notions in common usage.

1.$\mathcal{P}$ is called orthogonal if each pair of polynomials is orthogonal.

2. The definition of Dunkl and Xu.

The algebraically complete family $\mathcal{P}$ is called orthogonal if each polynomial p_n of degree $|n|$ is orthogonal to each polynomial q_m for which $|m| < |n|$. Further, for each m, n, such that $|m| = |n|$ define $s_{m,n} = \langle x^m, p_n \rangle$. Then the $|n| + 1 \times |n| + 1$ matrix $(s_{m,n})$ must be non-singular, i.e. $det(s_{m,n}) \neq 0$

The subspace of polynomials of degree $|n|$ are uniquely determined, but not the individual polynomials.

3. Biorthogonality

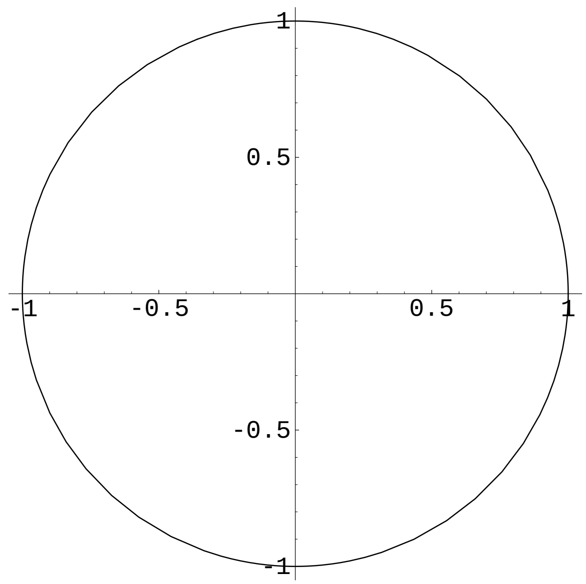
Given two algebraically complete families $\mathcal{P}$ and $\mathcal{Q}$. If both families have the property that $\langle p_m, p_n \rangle = 0$ if $|m| < |n|$, and further, if $p_m \in \mathcal{P}$ and $q_n \in \mathcal{Q}$ then $\langle p_m, q_n \rangle = 0$ if $|m| = |n|$ and $m \neq n$, then the families are called biorthogonal. (Appell and Kampé de Fériet)

Characters are orthogonal

Theorem. If p and q are characters of a compact hypergroup, then either $p = q$ or p is orthogonal to q. (Dunkl 73 and independently by others)

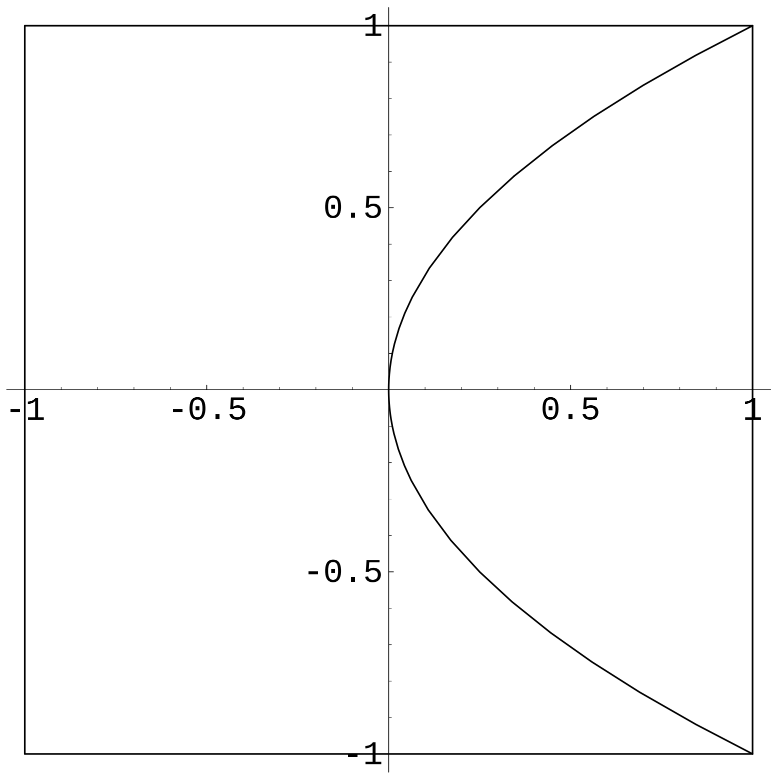
8. Tom Koornwinder's famous list

Classes I and II. Polynomials in the Disk



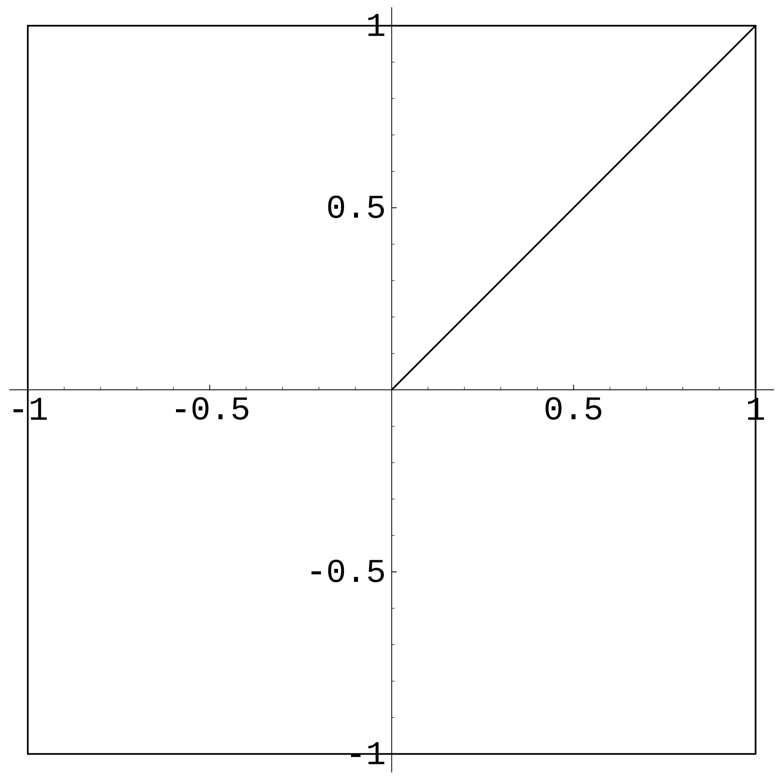
$$w(x) = (1 - x^2 - y^2)^\alpha$$

III. the Parabolic Biangle



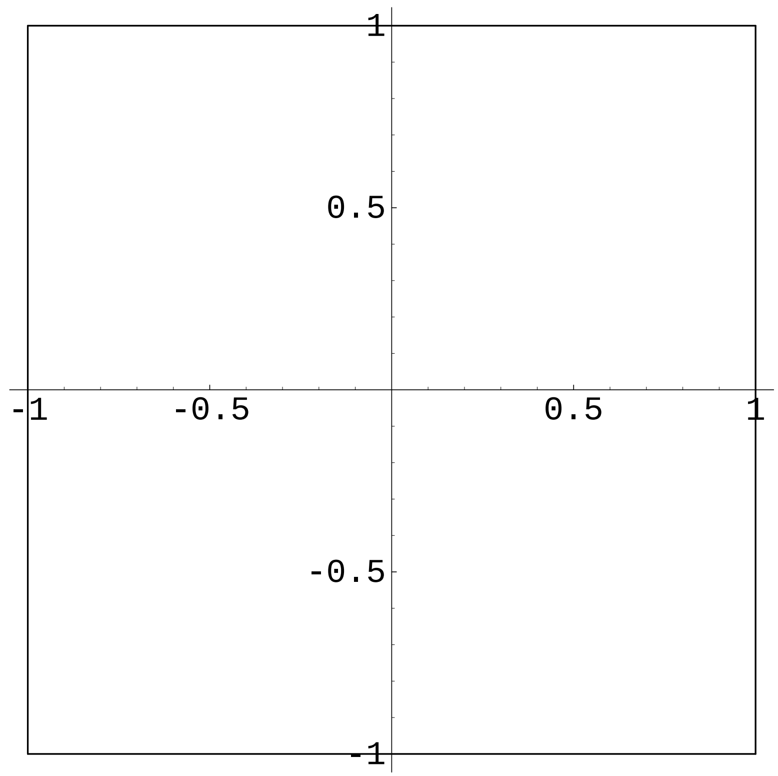
$$w(x) = (1 - x)^{\alpha}(x - y^2)^{\beta}$$

IV. The Triangle



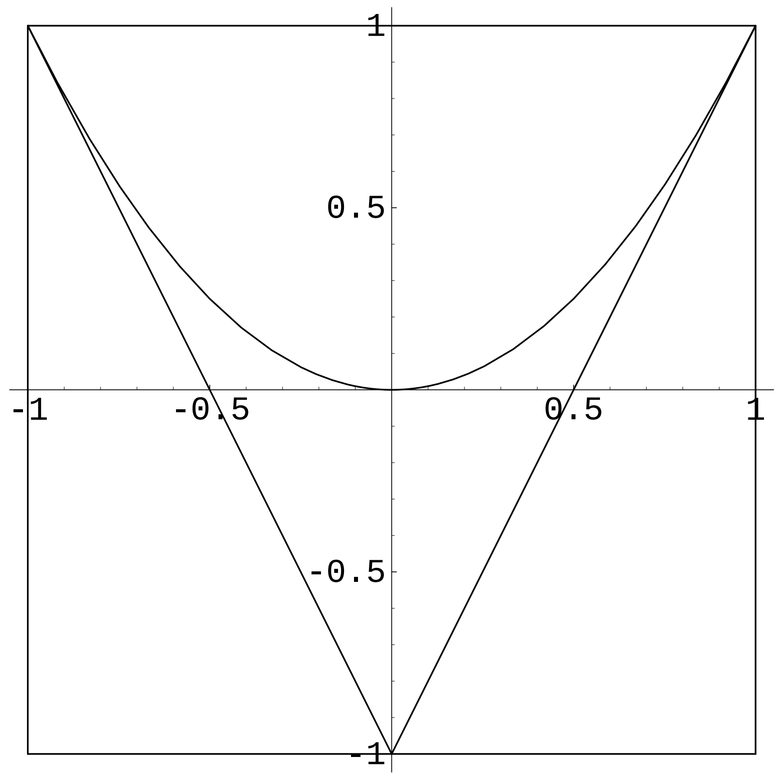
$$w(x) = (1-x)^{\alpha}(x-y)^{\beta}y^{\gamma}$$

V. The Square



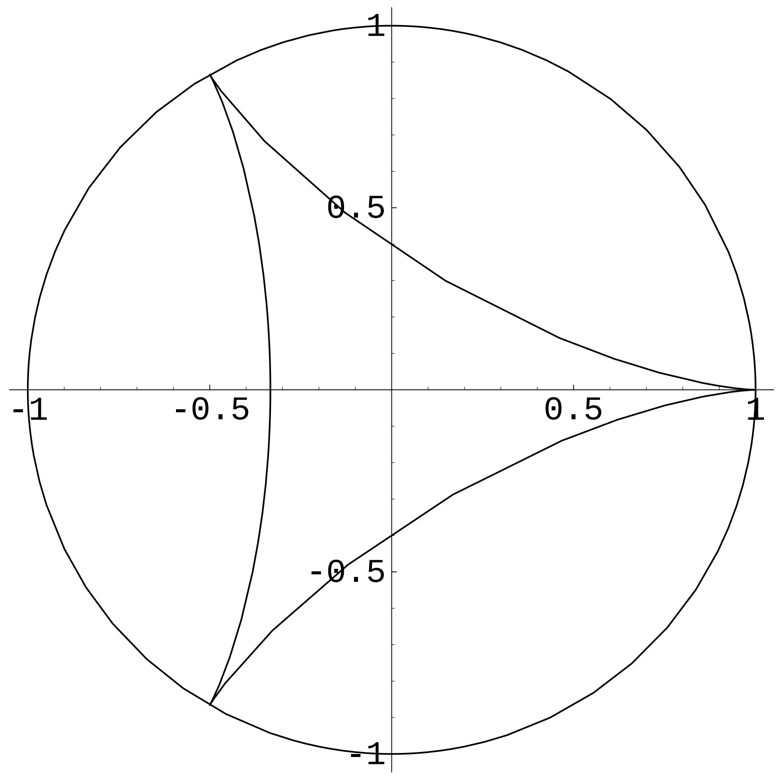
$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\gamma}(1+y)^{\delta}$$

VI. The Parabolic Triangle



$$w(x) = (1 - 2x + y)^{\alpha}(1 + 2x + y)^{\beta}(x^2 - y)^{\gamma}$$

VII. Steiner's hypocycloid



$$w(x) = [-3(x^2 + y^2 + 1) + 8(x^3 - 3xy^2) + 4]^{\alpha}$$

9. Hermitian polynomial hypergroups in two variables

1. Polynomials orthogonal in the Parabolic biangle (Class III).

Let

$$P_{n,k}^{\alpha,\beta}(x,y) = P_{n-k}^{\alpha,\beta+k+\frac{1}{2}}(2x-1)\, x^{\frac{k}{2}}\, P_k^{\beta,\beta}(x^{-\frac{1}{2}}y)$$

Koornwinder and Schwartz (KS95) showed that this case has a hypergroup product formula for all $\alpha \geq \beta + \frac{1}{2} \geq 0$

2. Polynomials orthogonal on the triangle (Class IV).

$$P_{n,k}^{\alpha,\beta,\gamma}(x,y) = P_{n-k}^{\alpha,\beta+\gamma+2k+1}(2x-1)\, x^k\, P_k^{\beta,\gamma}(2x^{-1}y-1)$$

These polynomials satisfy a hypergroup product formula for

$$\alpha \geq \beta+\gamma, \quad \beta \geq \gamma \geq -\frac{1}{2}$$

(KS95).

3. Products of Jacobi polynomials on the square (Class V).

Let

$$P_{n,k}^{\alpha,\beta,\gamma,\delta}(x,y) = P_{n-k}^{\alpha,\beta}(x) P_k^{\gamma,\delta}(y)$$

This family has a hypergroup product formula for all

$$(\alpha,\beta),(\gamma,\delta) \in E_J$$

4. Polynomials on the Parabolic Triangle (ClassVI).

The case $\gamma = -\frac{1}{2}$ can be obtained by symmetrizing Jacobi polynomials on the unit square, and then by a change of variables, $s = x+y$ and $t = xy$ we obtain an algebraically complete family of polynomials which derives its product formula from the product formula for the constituent Jacobi polynomials. This product formula is a hypergroup product formula for all $(\alpha, \beta) \in E_J$ and $\gamma = -\frac{1}{2}$. It is not known if there is a product formula for the other values of α, β, γ, much less one that leads to a hypergroup.

10. Non-Hermitian polynomial hypergroups in two variables

1. The Disk polynomials(Class I).

Let D_γ be the collection of polynomials of the form

$$\begin{aligned} P^{\gamma}_{m,n}(z) &= P_n^{\alpha,m-n}(2z\bar{z}-1)z^{m-n} \quad m \geq n \\ &= P_m^{\alpha,n-m}(2z\bar{z}-1)\bar{z}^{n-m} \quad m < n \end{aligned}$$

This family was shown to have a positive convolution for $\gamma \geq 0$ by Annabi and Trimeche (AT74) and Kanjin (Kan76) and others. These polynomials are the characters of a non-Hermitian hypergroup for $\gamma > 0$. See (CS91)and (GS95)

2. Another orthogonal family on the disk (Class II)

The collection of polynomials

$$P^{\alpha}_{n,k}(x,y) = P^{\alpha+k+\frac{1}{2},\alpha+k+\frac{1}{2}}_{n-k}(x)\,(1-x^2)^{\frac{k}{2}}\,P^{\alpha,\alpha}_{k}((1-x^2)^{-\frac{1}{2}}y)$$

have not been fully studied. If it has a product formula, it can not lead to a hypergroup because the first degree characters and the identity do not agree.

3. Polynomials orthogonal in Steiner's hypocycloid (Class VII)

Let $P^{\alpha}_{m,n}(z,\bar{z}) = z^m \bar{z}^n + p(z,\bar{z})$ where it is assumed that $deg(p) < m+n$, where p is chosen so that $P^{\alpha}_{m,n}$ is orthogonal to every polynomial of lower degree, and in fact can be turned into a family of orthogonal polynomials.

These polynomials have been studied extensively by Koornwinder. It is not known if this family has a product formula for other than some special geometric cases. It was shown that in the so called "Chebyshev" case of $\alpha = -\frac{1}{2}$, that there is a discrete product formula, and this product formula generates a non-Hermitian hypergroup.(Con03).

11. Conclusion

There is clearly much more to do. The basic classification scheme into Hermitian and non-Hermitian has been extended to higher dimensions. several of the examples here have ready generalizations to this setting, for example the simplex polynomials. On the other hand until we have a better understanding of which families in two dimensions have positive product formulas, we have little hope of a final theory.

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