An Introduction to Network Information Theory with Slepian-Wolf and Gaussian Examples

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What is Network Information Theory?

We have come to know that Information Theory is the branch of probability theory which includes the application of communication systems. This branch of mathematics and computer science was introduced by communication scientist studying statistical structures of electrical communication equipment. So again we pose the question, what is Network Information Theory? It involves the fundamental limits of communication and Information Theory in networks with multiple senders and receivers and optimal coding techniques and protocols which achieve these limits. It extends Shannon's point-to-point information theory to networks with several sources and destinations. An important goal is to characterize the capacity region or optimal rate which is the set rate of the ordered list of elements in which there exist codes with reliable transmissions. These rates of tuples are known to be achievable. Although a complete theory is yet to be developed and the characterization of the regions of capacity is generally a difficult problem there have been positive results for multiple classes of networks. Computer networks are examples of large communication networks.

Even within a lone computer there are various computers that talk to each other. These large networks coupled with the advent of the internet and supported by advancements in semiconductor technology, error correction, compression, computer science and signal processing revived an interest in a subject which was somewhat dormant through the period from the mid 1980's up to the mid 1990's. Since the mid 1990's there has been a large scale interest in the activities of this subject. Not only has there been progress made on past problems, there has also been work dealing with new network models, scaling laws and capacity approximations and fresh approaches to coding for networks and subjects intersecting information theory and networking.

In Networking Information Theory successive refinement of information, successive cancelation decoding, multiple description and network coding are some of the methodologies expounded and implemented in the real world of networks. A good example of a multi-users would consist of U stations or users, where U = 1, 2… u, wishing to communicate with a familiar satellite over a familiar channel, known as a multiple access channel. The questions posed are, what rates of communication are achievable simultaneously? How do the users cooperate with each other when sending information to receiver? What are the limitations of interference among the users placed on the total rate? There are satisfying answers for the above questions. Reversing the network we can consider another example, one television station sending information to U TV receivers. The questions that arise here are what rates of information are sent to the different receivers? How does the sender encode information meant for different receivers in a signal that is common?

The answers are only known in special cases for this contrast channel. There are also other channels to consider as special cases of general communication network consisting of N nodes (connection points trying to communicate with one another). Those channels are the relay channels, two-way channels and interference channels. For these channels there are only some answers to the questions regarding the coding strategies and communication rates. Non-deterministic sources are associated with some of the nodes in the network. If there are independent sources then the nodes sends independent messages. We also must allow the source to be dependent also.

This brings to light additional questions, with the channel transition function and the probability distribution; can we transmit these sources over the channel and recover the sources at the destination with suitable distortion? How can we beneficially use the dependence to diminish the sum of information transmitted? We will consider some of these network communication special cases. We first will look at the problem of source coding when the channels are noiseless and there is no interference. In these cases the problem is reduced to locating a set of rates that are associated with the sources in which the required sources are decoded at the destination with a low error probability.

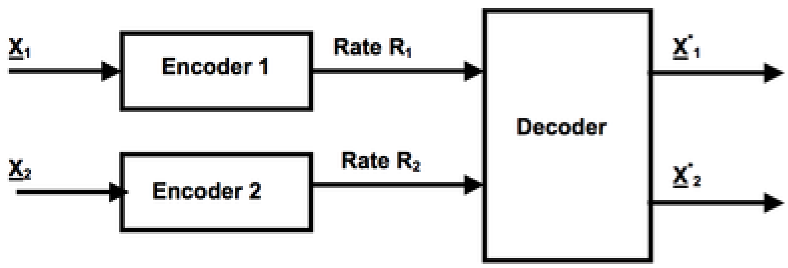
Slepian-Wolf

We now introduce the Slepian-Wolf source. Slepian and Wolf were two information theory researchers.

David Slepian **Jack K. Wolf**

David Slepian (June 30, 1923 – November 29, 2007) was an American mathematician born in Pittsburgh, Pennsylvania. Jack Kein Wolf (March 14, 1935 – May 12, 2011) was an American researcher in information theory and coding theory and was born in Newark, New Jersey. Slepian and Wolf worked together to discover a fundamental result in the distributed source coding.

The Slepian-Wolf source coding problem is the simplest case for source distribution coding. This involves having two sources that are separately encoded, but encoded at the common node. This example is shown in the figure below.



For two correlated streams, such systems employ the Slepian-Wolf coding which is a form of distributed source coding. Compared to an encoder that assumes that the data streams are independent, the separated encoders can achieve better compression rates by making use of the fact that the data systems are complementary. A surprising result is that the Slepian-Wolf coding can achieve the same compression rate as an optimal single encoder that has all correlated data streams as inputs. Even when the encoder has access to multiple correlated data streams, the Slepian-Wolf theorem has practical applications. For example, in order to diminish the complexity of image and video compression for cell phones, the stream may be encoded separately without reducing the compression rate.

Slepian-Wolf Theorem

The systems effectiveness is measured by the rates of encoded bits per source symbol of the compressed data streams which are outputs by the encoders. The Slepian-Wolf Theorem defines the set of rates that allows the decoder to reconstruct the correlated data streams with an arbitrarily small probability of error. Taking another look at the above figure, encoder 1, observes X_1 and sends a message to the decoder that is a number from the set {1, 2, 3,......... 2^{nR1}}. Encoder 2, which observes X_2, sends a message to the decoder that is a number set {1, 2…2^{nR2}}. The outputs from the two encoders are from the inputs to the single decoder. Upon receiving these two inputs, the decoder outputs two n-vectors X^*_1 and X^*_2 which are estimates of X_1 and X_2. We are interested in those systems which the probability X^*_1 does not equal X_1 or X^*_2 does not equal X_2 can be made small as desired by choosing a sufficiently large *n*. This system is known as an admissible system and the rate pair (R_1, R_2) for and admissible system is an admissible rate pair. The set closure of all the admissible rate pairs is known as the admissible rate region. We can obtain the following entropies for the pair of variables X_1 and X_2 with joint probability distributions of P ($X_1 = x_1$, $X_2 = x_2$):

$$H(X_1, X_2),\ H(X_1|X_2),\ H(X_2|X_1),\ H(X_1) \text{ and } H(X_2)$$

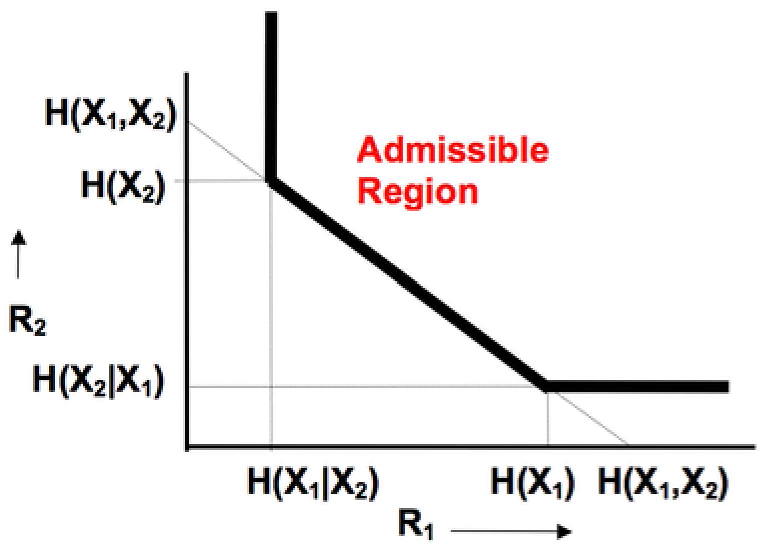
The admissible rate region, which we have indicated to be the set closure of all admissible rate pairs, is the set of points (R1, R2) satisfying three inequalities:

$$R_1 \geq H(X_1 \mid X_2) +$$

$$R_2 \geq H(X_2 \mid X_1)$$

$$R_1 + R_2 \geq H(X_2, X_1)$$

The admissible rate region is shown in the figure below.

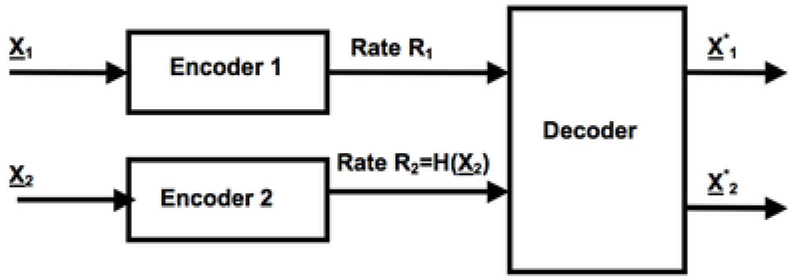


The importance of the Slepian-Wolf Theorem is realized by the comparison with the entropy bound for single source compression. Separated encoders that ignore the source correlation achieve rates only of $R_1 + R_2 \geq H(X_1) + H(X_2)$. Yet with the Slepian-Wolf coding, the separated encoders are able to achieve their knowledge of the correlation to accomplish the same rates as an optimal joint encoder, $R_1 + R_2 \geq H(X_1, X_2)$.

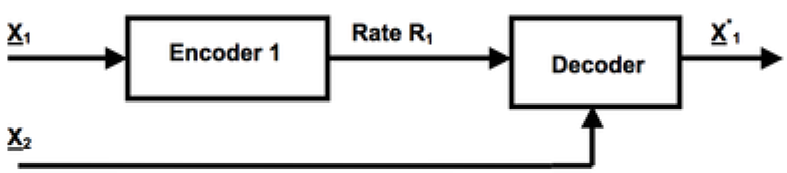
Slepian-Wolf Theorem Proof

The condition of the aforementioned three inequalities follows by considering a system change where the source pair sequences, X_1 and X_2 are inputs to a separated encoder. The separated encoder's output rate must at least equal $H(X_1, X_2)$, giving: $R_1 + R_2 \geq H(X_1, X_2)$.

If the encoder knows X_1 and X_2 and the decoder also knows X_2, the encoder will need a code rate at least H $(X_1|X_2)$ giving $R_1 \geq H(X_1 | X_2)$. The remaining inequality, $R_2 \geq H(X_2 | X_1)$ follows symmetrically. Showing the adequacy of the three inequalities we consider the rate pair, $R_1 = H(X_1|X_2)$, $R_2 = H(X_2)$ on the boundary region of the admissible rate region. If $R_2 = H(X_2)$ then the output of encoder 2 satisfies the reconstructed X_2, so the block diagram shown in the below figure



reduces to the diagram shown in the figure below



The initial construction of the admissible system at the rate point $R_1 = H(X_1|X_2)$, $R_2 = H(X_2)$ was determined for the statistical model of the correlated source pair, called the twin binary symmetric source. A twin binary symmetric source is a memory less source with outputs X_1 and X_2. These outputs are binary random variables with values 0 and 1 represented by:

$$P(X_1 = 0) = P(X_1 = 1) = 1/2,$$

$$P(X_2 = 0| X_1 = 1) = P(X_2 = 1 | X_1 = 0) = p,$$

$$P(X_2 = 0| X_1 = 0) = P(X_2 = 1|X_1 = 1) = (1 - p)$$

where p is the parameter satisfying $0 \leq p \leq 1$. We see that,

$$P(X_2 = 0) = P(X_2 = 1) = 1/2$$

Defining $h_2(p) = -[p \log_2(p) + (1 - p) \log_2(1- p)]$.

For the twin binary symmetric source we obtain:

$$H\ (X_1) = 1,$$

$$H\ (X_2) = 1,$$

$$H\ (X_2|X_1) = h_2\ (p),$$

$$H\ (X_1|X_2) = h_2\ (p),$$

$$H\ (X_1, X_2) = 1 + h_2\ (p).$$

For the twin binary symmetric source, the rate point of interest has $R_1 = H\ (X_1|X_2) = h_2\ (p)$ and $R_2 + H(X_2) = 1$. To resolve the problem of compressing X_1 we can think of the twin binary symmetric source model as if X_1 were passes through a (BSC) binary symmetric channel with a bit error probability, p, to obtain X_2. For large *n*, a parity code exists for the BSC with approximately $2^{n\ (1-h2\ (p))}$ code words. A decoder that sees the output channel, X_2, will be able to tell which code word was at the channel's input.

The problem that arises when applying this idea to the source code problem is that the input to the channel, X_1does not have to be one of the $2^{n(1-h2(p))}$ code words of the parity check code since X_1 can be any of the 2^n binary *n* -vectors. Another idea is necessary which comes from the fact that the co-set decomposition of the group of 2^n binary *n* -vectors in terms of the subgroup code words. A co-set of this subgroup is formed by taking various binary *n* -vector that is not in the subgroup and adding it bit by bit, mod 2 , to each vector in the subgroup to form a new set of $2^{n(1-h2(p))}$ vectors.

Repeat the process choosing a vector to be added, a binary vector that has not been added before, from either the original subgroup or any previous constructed co-sets. The process is completed when all 2^n binary vectors have surfaced in either the original group or in the co-sets. The co-sets are now either identical or disjoint and every binary *n* -vector appears in only one co-set when considering the subgroup as a co-set.

Given a code block of length n with $2^{n\ (1-h2\ (p))}$ code words, we have $2^{nh2\ (p)}$ co-sets since,

$$2^{nh2\ (p)} = 2^n/2^{n\ (1-h2\ (p))}.$$

Also the set of vectors in each co-set have similar error correction properties as the original linear code since the vectors in any co-set are translated versions of the original code words. These codes are called the co-set code.

After relating this to the problem, X_1 must be in one of the co-sets of the code group. When the source encoder transmits to the decoder the identity of the co-set which includes X_1, the decoder can locate X_1 from this knowledge and the knowledge of X_2 by using a decoder for the code's co-set that worked on the received X_2 word. Because there are $2^{nh2\ (p)}$ co-sets, the encoder transmits $n\,h2$ (p) binary digits. The rate of transmission is h_2 (p) = H ($X_1|X_2$). The admissibility of the rate point equals:

$$R_1 = H\ (X_1|X_2) = h_2\ (p) \text{ and } R_2 = H(X_2) = 1.$$

Now the complete admissible rate region follows from time-sharing, wasted bits and symmetry. We will next consider some Gaussian examples of basic channels of Network Information Theory.

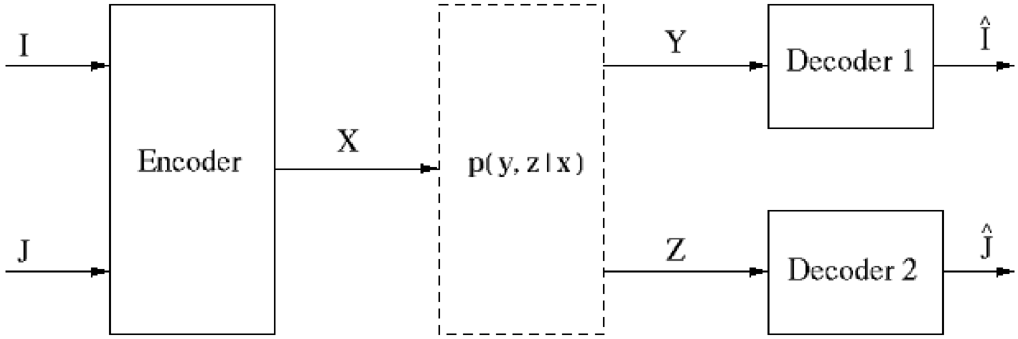
Gaussian Broadcast Channels

The concept of Gaussian processes is named after Carl Friedrich Gauss because it is based on the notion of the normal distribution.

Carl F. Gauss

Carl Gauss (April 30, 1777 – February 23, 1855) was a German mathematician and physical scientist who contributed to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy (a branch of applied mathematics and earth science), geophysics, electrostatics, astronomy and optics. He was sometimes referred to as the prince of mathematicians, or the foremost mathematician and the greatest mathematician since antiquity. Gauss is ranked as one of our history's most influential mathematicians. He referred to mathematics as the queen of science.

We will now define the broadcast channel; the broadcast channel is a communication channel which there is one sender and two or more receivers. The figure below illustrates the broadcast channel.



The simplest example of a broadcast channel would be a radio or television station. The station wants everyone tuned in to receive the same information. The capacity is Max $_{p(x)}$ and $\text{Min}_i\, I(X; Y_i)$. This may be less than the capacity of the worst receiver. The information can be arranged so that better receivers will receive additional information, thereby producing a better sound and picture. The worse receiver will continue to receive more basic information.

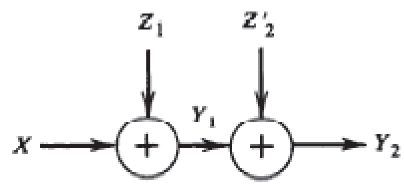
Since the introduction of High Definition TV it may be required to encode information so that poor receivers will receive regular TV signals and better receivers will receive the additional information to obtain the High Definition signal. We assume we have a sender with a power of P and two receivers, one with Gaussian noise of power N_1 and the other with Gaussian noise of power N_2. We also assume $N_1 < N_2$, so receiver Y_1 is less noisy than receiver Y_2. The model for this channel is $Y = X + Z_1$ and $Y_2 = X + Z_2$, where Z_1 and Z_2 are correlated Gaussian random variables with a variance of N_1 and N_2 respectively. All Gaussian broadcast channels belong to the degraded broadcast channel class. The capacity region of the Gaussian broadcast channel is $Y_i = X_i + Z_i$, where i =1, 2, 3… and where **Z** are Gaussian random variables with variance N and a mean equal to 0. The signal $\mathbf{X} = (X_1, X_2, \ldots X_n)$. The power constraint equals:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \leq P .$$

The Shannon capacity **C** is determined by maximizing **I** (x, y) over all the random variables **X** such that $EX^2 \leq P$ and is given by,

$$\mathbf{C} = \tfrac{1}{2} \log (1 + P/N) \text{ bits per transmission.}$$

The Gaussian broadcast channel is illustrated in the below figure.



One output is a degraded version of the other output. All Gaussian broadcast channels are equal to this type of degraded channel,

$$Y_1 = X + Z_1,$$

$$Y_2 = X + Z_2 = Y_1 + Z'_2,$$

where $Z_1 \sim N(0, N1)$ and $Z'_2 \sim N(0, N_2 - N_1)$. The capacity region of this channel is given by:

$$R_1 < C(\alpha P / N_1) \quad \text{and} \quad R_2 < C((1-\alpha) P / \alpha P + N_2)$$

where α equals ($0 \leq \alpha \leq 1$).

Converse for Gaussian Broadcast Channel

Since the Gaussian Broadcast Channel's capacity region is the same as the physically degraded Gaussian Broadcast Channel, we can prove the converse for the physically degraded Gaussian Broadcast Channel. Using Fano's inequality (also known as Fano converse and the Fano lemma, relates the average information lost in a noisy channel to the probability of the categorized error. Derived by Robert Fano professor emeritus of Electrical Engineering and Computer Science at Massachusetts Institute of Technology).

$$nR_1 \leq I(M_1; Y^n{}_1 | M_2) + n\,\varepsilon_n,$$

$$nR_2 \leq I(M_2; Y^n{}_2) + n\,\varepsilon_n,$$

We next need to show that there exist an $\alpha \in [0, 1]$ such that

$$I(M_1 : Y^n{}_1 | M_2) \leq nC(\alpha S_1) = nC(\alpha P / N_1)$$

and I $(M_2; Y^n{}_2) \leq nC(\alpha S_2 / \alpha S_2 + 1) = nC(\alpha P / \alpha P + N_2)$,

Consider

$$I(M_2; Y^n{}_2) = h(Y^n{}_2) - h(Y^n{}_2|M_2) \leq n/2 \log(2 \pi e (P + N_2)) - h(Y^n{}_2 | M_2)$$

Since

$n/2 \log(2 \pi e N_2) = h(Z^n{}_2) = h(Y^n{}_2|M_2, X^n) \leq h(Y^n{}_2 | M_2) \leq h(Y^n{}_2) \leq n/2 \log(2 \pi e (P + N_2))$,

there must exist an α Ɛ [0, 1] such that

$$h(Y^n{}_2|M_2) = n/2 \log(2 \pi e (P + N_2)). *$$

Next we consider

$$I(M_1; Y^n{}_1|M^2) = h(Y^n{}_1|M_2) - h(Y^n{}_1|M_1, M_2)$$

$$= h(Y^n{}_1|M_2) - h(Y^n{}_1|M_1, M_2, X^n)$$

$$= h(Y^n{}_1|M_2) - h(Y^n{}_1| - h(Y^n{}_1)$$

$$= h(Y^n{}_1|M_2) - n/2 \log(2 \pi e N_1).$$

Now using the conditional entropy we obtain

$$h(Y^n{}_2|M_2) = h(Y^n{}_1 + Z^n{}_2 | M_2)$$

$$\geq n/2 \log(2^{2h(Yn1 \mid M2)/n} + 2^{2h(Zn2 \mid M2)/n)}$$

$$= n/2 \log(2^{2h(Yn1 \mid M2)/n} + 2 \pi e (N_2 - N_1).$$

Combining this inequality with the above equation marked with a * implies that

$$(2 \pi e (\alpha P + N_2) \geq 2^{2h(Yn1 \mid M2)/n} + 2 \pi e (N_2 - N_1).$$

Thus, $h(Y^n{}_1|M_2) \leq (n/2)\log(2 \pi e (P + N_1))$ and hence

$$I(M_1; Y^n{}_1|M_2) \leq n/2 \log(2 \pi e (\alpha P + N_1)) - n/ \log(2 \pi e N_1) - nC(\alpha P / N_1))$$

This completes the proof.

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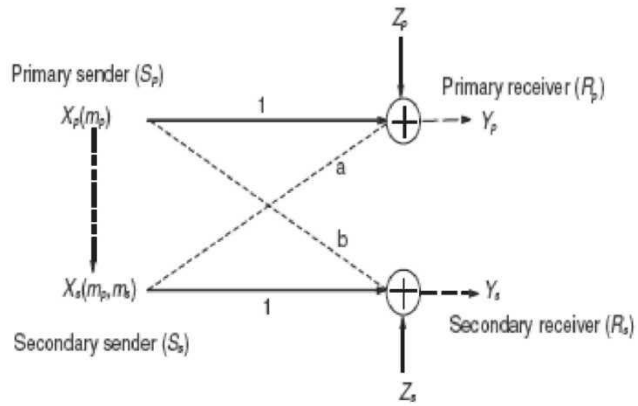
Gaussian Interference Channels

There are two senders and two receivers with the Gaussian interference channel. Sender 1 wishes to send information to receiver 1. Sender 1 does not care what receiver 2 receives. The same holds true for sender 2 and receiver 3. Each channel interferes with one another. The channel is illustrated in the below figure.



Since there is only one receiver for each sender it is not quite a broadcast channel, nor is it a multiple access channel since each receiver is interested only in what is being sent by the similar transmitter. We obtain a symmetric interference of,

$$Y_1 = X_1 + aX_2 + Z_1$$

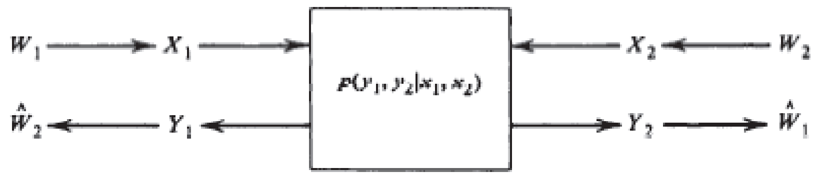
$$Y_2 = X_2 + aX_1 + Z_2$$

where Z_1 and Z_2 are both independent random variables $\mathcal{N}(0, N)$. This is one channel that has not been generally solved even in the Gaussian case. But in the case of high interference, it can be remarkably shown that the region of the capacity of this channel is the same as if there were no interference at all. To achieve this, two code books are generated, each having a power of P and a rate of C (P / N). Each sender then chooses a word from his book and sends it. Now given interference a, which satisfies $C(a^2 P / (P + N)) > C(P / N)$ then the first transmitter understands perfectly the index of the second transmitter. The index is found by looking for the code word closest to his received signal.

Once the signal is found it is subtracted from the received waveform that now presents a clean channel between the first sender and the second sender. The sender's code book is searched to locate the closest code word which is then declared the code word that was sent.

Gaussian Two Way Channel

The only difference between the interference channel and the two way channel is that the two way channel's sender1 is attached to receiver 2 and sender 2 is attached to receiver 1. This is shown in the below figure.



This allows sender1 to use information from receiver 2 symbols previously received to determine what to send next. The two way channel also introduces another fundamental condition of Network Information Theory. This condition is called feedback. Feedback allows the sender to use limited information that each has about the other message to concur with one another. The two way channel's capacity region is not known in general, although Shannon obtained upper and lower bounds of the region. These two bounds coexist for Gaussian channels and the region's capacity is known. The Gaussian two-way channel separates into two independent channels. We let the powers of transmitters1 and 2 equal P_1 and P_2 and the noise variances of the two channels equal N_1 and N_2. Then the rates are equal to,

$$R_1 < C\ (P_1 / N_1)$$

$$R_2 < C\ (P_2 / N_2)$$

this can be accomplished by the methods of the interference channel. So with this case we would generate two code books with rates R_1 and R_2. Sender1 sends a code word from code book 1. Receiver 2 receives the sum of the code words sent by two senders plus some noise. Receiver 2 simply cuts out sender 2 code word which gives him a clean channel from sender1 (with only the variance N_1 noise). So the two way Gaussian channel separates into two independent Gaussian channels. This is not the general case of the two way channel; in general there will be a tradeoff between the two senders so that the both of them cannot send optimal rates simultaneously.

Bibliography

Wolf, K. Jack and Kukoski, M. Brian. "Slepian-Wolf coding." 21 October 2011, 04:16 <http://www.scholarpedia.org/article/Slepian-Wolf_coding>

Cover, M. Thomas. Elements of Information Theory. Chapter 14 Network Information Theory. New York: Wiley-Interscience; 26 August, 1991

"Gaussian Process". Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 4 May 2013 at 13:16. Web. 3 May.2012

Gama, E. Abbas and Kim, H. Young. Network Information Theory. New York: Cambridge University Press ; January 16, 2012

"Robert Fano". Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 31 March 2013 at 21:09.. Web. 3 May.2013

"David Slepian". Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 07 March 2013 at 21:09. Web. 3 May.2013

"Jack K. Wolf". Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 17 February 2013 at 21:09.. Web. 3 May.2012

"Carl F. Gauss". Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 04 May 2013 at 21:09.. Web. 3 May.2012